

KAZHDAN PROJECTIONS, RANDOM WALKS AND ERGODIC THEOREMS

CORNELIA DRUȚU AND PIOTR W. NOWAK

ABSTRACT. In this paper we investigate generalizations of Kazhdan's property (T) to the setting of uniformly convex Banach spaces. We explain the interplay between the existence of spectral gaps and that of Kazhdan projections. Our methods employ Markov operators associated to a random walk on the group, for which we provide new norm estimates and convergence results. This construction exhibits useful properties and flexibility, and allows to reclassify the existence of Kazhdan projections from a C^* -algebraic phenomenon to a natural object associated to random walks on groups.

We give a number of applications of these results. In particular, we address several open questions. We give a direct comparison of properties (TE) and FE with Lafforgue's reinforced Banach property (T) ; we obtain shrinking target theorems for orbits of Kazhdan groups and apply them to a question of Kleinbock and Margulis; finally, we construct non-compact ghost projections for warped cones, answering a question of Willett and Yu. In this last case we conjecture that such warped cones provide new counterexamples to the coarse Baum-Connes conjecture.

1. INTRODUCTION

One way to investigate properties of groups, especially with a view to their actions on Banach spaces, is through the group Banach algebras. These are natural analytic objects encoding many properties of the group. The existence of projections in such algebras is a particularly important and challenging problem. For instance, the (non)existence of idempotents other than 0 and 1 in the reduced group C^* -algebra of a torsion-free group is a long-standing conjecture of Kadison and Kaplansky. When the group is amenable (and more generally, a-T-menable) the Kadison-Kaplansky conjecture is known to be true. Additionally, for amenable and torsion free groups the maximal group C^* -algebra is isomorphic to the reduced group C^* -algebra, therefore the maximal group C^* -algebra does not have non-trivial idempotents either.

Date: October 13, 2015.

The research of both authors was supported by the EPSRC grant "Geometric and analytic aspects of infinite groups". The research of the first author was also partially supported by the project ANR Blanc ANR-10-BLAN 0116, acronym GGAA, and by the Labex CEMPI (ANR-11-LABX-0007-01). The research of the second author was partially supported by Narodowe Centrum Nauki grant 2013/10/EST1/00352.

A main result of this paper is an explicit construction of proper idempotents in many group Banach algebras. The construction is based on random walks, a new ingredient in this setting. Our construction turns out to be relevant in various contexts, from expander graphs to ergodic geometry and the Baum-Connes conjecture.

A *Kazhdan projection* for a locally compact group G is an idempotent in the maximal group C^* -algebra $C_{\max}^*(G)$, whose image under any unitary representation is the projection onto the space of invariant vectors. Such projections exist in $C_{\max}^*(G)$ if and only if the group G has Kazhdan's property (T) [1]. They are important for many applications. A classical consequence of their existence is the fact that the map on K -theory induced by the canonical homomorphism $C_{\max}^*(G) \rightarrow C_r^*(G)$ from the maximal to the reduced group C^* -algebra, fails to be an isomorphism for Kazhdan groups, see e.g. [18, Ch. 2, S. 4]. They play the main role in the failure of some versions of the Baum-Connes conjecture, since projections of this type cannot be in the image of the Baum-Connes assembly map [27]. Kazhdan projections are the main ingredient of Lafforgue's reinforced Banach property (T) and allowed for the construction of the first examples of expanders with no coarse embedding into any uniformly convex Banach space [35]. Finally, they also play an important role in the generalization of property (T) to C^* -algebras [11].

However, Kazhdan projections have always been considered somewhat mysterious objects, whose existence can eventually be ascertained, but who cannot be constructed explicitly (see e.g. [28, Footnote 22, page 73]).

Spectral gaps, Markov operators and projections. At the core of our paper is a study of a Banach space version of property (T), formulated in a very general setting: with respect to a given family of isometric representations on Banach spaces. We prove that such a property can be characterized in three different ways: the standard spectral gap property, the behavior of the Markov operator on a canonical complement of the fixed vectors subspace, and the existence of a Kazhdan projection, with an explicit formula to calculate it, using Markov operators.

Indeed, given an isometric representation π of a group G on a reflexive Banach space E , the subspace E^π of fixed vectors has a canonical π -complement, E_π (see Section 2.c for details). Given a probability measure μ on G , let A_π^μ denote the Markov (averaging) operator associated to π via μ . We prove the following.

Theorem 1.1. *Let G be a locally compact group, and \mathcal{F} a family of isometric representations of G on a uniformly convex family \mathcal{E} of Banach spaces. The following conditions are equivalent:*

- (i) *the family \mathcal{F} has a spectral gap (see Definition 2.3);*
- (ii) *there exists a compactly supported probability measure μ on G and $\lambda < 1$ such that for every isometric representation $\pi \in \mathcal{F}$ of G on $E \in \mathcal{E}$ we have $\|A_\pi^\mu|_{E_\pi}\| < \lambda$;*

- (iii) *there exists a compactly supported probability measure μ on G and a number $\mathfrak{S} < \infty$ such that for every $\pi \in \mathcal{F}$ the iterated Markov operators $(A_\pi^\mu)^k$ converge with speed summable to at most \mathfrak{S} to the projection \mathcal{P}_π onto E^π along E_π , that is*

$$\left\| (A_\pi^\mu)^k - \mathcal{P}_\pi \right\| \leq a_k,$$

where $\sum_k a_k \leq \mathfrak{S}$.

In Theorem 3.8 we give an explicit formula for the projection \mathcal{P}_π in terms of the Neumann series of the Markov operator:

$$(1) \quad \mathcal{P}_\pi = I_E - \left(\sum_{n=0}^{\infty} (A_\pi^\mu)^n \right) (I_E - A_\pi^\mu).$$

The hypothesis of uniform convexity is needed only in the implication (i) \Rightarrow (ii) and (iii), for the other implications it suffices to have a family of complemented representations on Banach spaces, in the sense of Definition 2.2.

When G has Kazhdan's property (T) and E is a Hilbert space, Theorem 1.1 holds for \mathcal{F} the family of all unitary representations of G . However, as Theorem 1.1 is formulated in terms of a family of representations, it also applies in the setting of Property (τ) (see Section 5 and the corresponding paragraph later in the Introduction), of property ($T\ell_p$) introduced in [5] etc.

The equivalence in Theorem 1.1 has an effective side to it, described below. Given a Kazhdan pair (Q, κ) defining the spectral gap (see Definition 2.3), the conditions (ii) and (iii) hold for a large class of measures, which we call *admissible with respect to the Kazhdan set Q* , explicitly constructed by means of Q , see Definition 2.1. For every such measure μ , a constant λ as in (ii) can be computed using the Kazhdan constant κ , the modulus of uniform convexity of the family \mathcal{E} and the choice of an appropriate compactly supported function on G associated to μ . Property (iii) then holds with $a_k = \lambda^k$. Conversely, given a measure μ and $\lambda \in (0, 1)$ satisfying either (ii) or (iii) with $a_k = \lambda^k$, the support of μ is a Kazhdan set and its corresponding Kazhdan constant is $1 - \lambda$. This implication applies, for instance, in the case of semisimple Lie groups with finite center, and their unitary representations on Banach spaces, to any probability measure with symmetric support not contained in a closed amenable subgroup [59, Theorem C].

One of the advantages of Theorem 1.1 is the high degree of flexibility in ensuring uniformity of several parameters for classes of isometric representations. For instance, in the case of groups admitting finite Kazhdan sets (see Section 3.g) this uniformity depends only on three quantities:

- (a) the Kazhdan constant of the family of representations,
- (b) the cardinality of the Kazhdan set Q ,
- (c) the modulus of uniform convexity of the Banach spaces.

It does not even depend on the group G , as long as we can arrange the above three items to have uniform bounds.

For applications, the existence of finite Kazhdan sets is a considerable asset: the averages become finite, the random walks discrete and an algorithmical approach and the use of computer become possible (see for instance Theorem 3.8, Remark 3.10 and Section 6). As it turns out, the existence of such finite sets is ensured in many cases outside the class of finitely generated groups, and in many cases the sets are described explicitly, as explained briefly in Section 3.g.

Kazhdan projections in group Banach algebras and Lafforgue's reinforced Banach property (T). The uniform convergence described in Theorem 1.1, (iii), depending on the Kazhdan constant, the modulus of uniform convexity of the family \mathcal{E} , and the choice of the measure μ , shows that the existence of a Kazhdan projection in group Banach algebras is a consequence of a uniform version of property (TE). Property (TE) was introduced and studied in [21, 2] as a natural generalization of property (T) from Hilbert to Banach spaces.

Theorem 1.2 (see Theorem 4.6 and Corollary 4.7). *Let G be a locally compact group and let \mathcal{F} be a family of isometric representations of G on a uniformly convex family of Banach spaces. There exists a Kazhdan projection $p \in C_{\mathcal{F}}(G)$ if and only if the family \mathcal{F} has a spectral gap.*

In particular, if G has Kazhdan's property (T) then there exists a Kazhdan projection in the L_p -maximal group algebra $C_{\max}^p(G)$ for every $1 < p < \infty$.

Here, $C_{\mathcal{F}}(G)$ is a natural version of the maximal C^* -algebra of G for the family \mathcal{F} of representations, see Definition 4.1. Banach group algebras for larger than isometric classes of representations were introduced and studied by V. Lafforgue [35]. We point out that Theorem 1.2 gives an entirely new proof of the existence of a Kazhdan projection even in the classical case of property (T) and Hilbert spaces. Only two previous proofs are known. The first is due to Akemann and Walter [1] and it relies on positive definite functions, a tool available essentially only in Hilbert spaces. Another proof, using minimal projections in C^* -algebras, is due to Valette [63]. The topic of operator algebras on L_p -spaces is an emerging direction in non-commutative geometry. Additionally, an approach to the Novikov conjecture via L_p -versions of the Baum-Connes conjecture has been recently developed by Kasparov and Yu. Theorem 1.2 shows that for groups with property (T) the same obstructions as in the Hilbert space case are likely to exist in K -theory.

Theorem 1.1 allows to compare V. Lafforgue's definition of reinforced Banach property (T) [35] to other generalizations of property (T) to Banach spaces, i.e. properties (TE) and FE [21, 2]. The question of such a comparison has been considered by several experts previously. In [36] it was shown that the reinforced Banach property (T) implies FE, and in [2] it was shown that property FE implies (TE). Since Lafforgue's reinforced Banach property (T) is formulated in terms of existence of Kazhdan projections in certain group Banach algebras, Theorem 1.2 provides implications in the other direction. We discuss this in detail in Section 4.b.

Property (τ) and expanders in the Banach setting. As Theorem 1.1 holds for a family of representations, it can be applied in the context of property (τ) , introduced by Lubotzky. Thus, we use Theorem 1.1 to formulate a complete, canonical generalization of property (τ) to uniformly convex Banach spaces (This contrasts with the situation for property (T) , for which there are several competing generalizations). Our generalized property (τ) is moreover consistent with a notion of expanders for Banach spaces, defined using Poincaré inequalities (see Definition 5.2).

More precisely, for a uniformly convex Banach space E we introduce property (τ_E) by the same definition as for Hilbert spaces, requiring that certain isometric representations factoring through finite quotients of G are separated from the trivial representation; that is, they have a uniform spectral gap. The following is a consequence of Theorem 1.1.

Theorem 1.3. *Let E be a uniformly convex Banach space, let G be a finitely generated residually finite group and let $\mathcal{N} = \{N_i\}$ be a collection of finite index subgroups with trivial intersection. The following conditions are equivalent:*

- (i) G has property (τ_E) with respect to $\mathcal{N} = \{N_i\}$ and a symmetric Kazhdan set Q ;
- (ii) the Cayley graphs $\text{Cay}(G/N_i, Q)$ form a sequence of E -expanders;
- (iii) there exists a Kazhdan projection $p \in C_{\mathcal{N}(E)}(G)$.

In the Hilbert space case the algebra appearing in the condition (iii) is a C^* -algebra. Note that the Kazhdan set Q in Theorem 1.3 does not necessarily generate G . Examples of Kazhdan sets that are not generating exist already for groups G and collections \mathcal{N} having the classical property (τ) . For instance, if $G = SL_n(\mathbb{Z})$, a finite symmetric set generating a subgroup Zariski dense in $SL_n(\mathbb{R})$ is a Kazhdan set for an appropriate choice of \mathcal{N} (see [9] and references therein). See also [58] for an earlier example of non-generating Kazhdan set for actions on expanders that are finite quotients.

Applications to ergodic theory. Another area in which Theorem 1.1 finds natural applications is ergodic theory. Consider, for instance, a group with property (T) acting ergodically on a probability space (X, ν) , and let f be an arbitrary function in $L^2(X)$. If the two operators appearing in the equality (1) are applied to f , then the left hand side becomes $\int_X f d\nu$, and the entire formula becomes a Birkhoff-type theorem, in which an exact explicit formula is provided, instead of just an estimate for the remainder.

Thus, a striking consequence of our results is that, while for ergodic actions of amenable groups the best way to average is *via* sequences of Følner sets, for ergodic actions of groups with property (T) a most effective averaging is *via* sequences of measures with compact support approximating the Kazhdan projection.

Generalizations to group actions of ergodic theorems of Birkhoff and von Neumann have been an object of significant interest, see [12, 43] for a survey and history of this subject. An early result of this type is the classical ergodic

theorem of Oseledec [50], that the time averages of a function over convolution powers of a probability measure converge to the mean of the function over the space. Theorem 1.1 allows to achieve a much stronger type of convergence, in norm topology instead of the strong operator topology, and with uniform estimates, depending on the spectral gap. It also allows to average using measures with finite support (equal to a Kazhdan set) even in the case of non-discrete groups (see Section 3.g).

Theorem 1.4. *Let G be a locally compact group and let $(X_i, \nu_i), i \in I$, be a family of probability spaces endowed with measure preserving ergodic actions of G . Consider also a collection \mathcal{E} of uniformly convex Banach spaces, and a number p in $(1, \infty)$.*

Assume that a family \mathcal{F} of isometric representations of G on $L_p(X_i, \nu_i; E)$, with $i \in I$ and $E \in \mathcal{E}$, induced by the measure preserving actions of G , has a spectral gap and let (Q, κ) be a Kazhdan pair.

For every Q -admissible measure μ on G there exists $\lambda < 1$, depending only on p , the normalizing factor of μ , the modulus of convexity of \mathcal{E} , and κ , such that

$$(2) \quad \left\| A_\pi^{\mu^k} f - \int_X f d\nu \right\|_p \leq \lambda^k \|f\|_p.$$

Proof. Since $\mathcal{P}_\pi = \int_X f d\nu$, the assertion follows from Theorem 1.1 (or, more precisely, from Theorem 3.4). \square

Theorem 1.4 becomes very concrete in certain cases. When G has property (T) and $E = \mathbb{R}$, we obtain a uniform quantitative ergodic theorem for *all* probability preserving actions of G and for any fixed $1 < p < \infty$, see Theorem 6.2. Another case is when $G = SL_3(F)$ for F a non-archimedean local field. It follows from [35] that for any uniformly convex E and any $1 < p < \infty$ the family of isometric representations of G on $L_p(X, \nu; E)$ has a spectral gap. Consequently, the above theorem holds for such G and any fixed E and p as above.

Another type of problems to which Theorem 1.1 can be applied are *shrinking target problems*, which ask how often does a typical orbit of an action hit a sequence of shrinking subsets. This problem for orbits of cyclic (or uniparameter) groups in locally symmetric spaces, and for shrinking sequences of neighborhoods of a cusp, has been thoroughly investigated and answered by Sullivan [62] and Kleinbock and Margulis [32]. The latter formulated the problem of finding similar results for shrinking sequences of neighborhoods of a point. Although some estimates are known in the case of rank 1 locally symmetric spaces, the problem is completely open in the case of higher rank. Theorem 1.1 allows to provide quantitative estimates in terms of random walks for the behavior of an ergodic action of a group with property (T) with respect to a shrinking target. For instance, we have the following theorem (we refer to Section 6.b for details, stronger statements and other corollaries).

Theorem 1.5 (see Theorem 6.6). *Let G be a locally compact group, and Γ a lattice in it. Let $\{\Omega_n\}$ be a sequence of measurable subsets in G/Γ .*

Assume that a locally compact group Λ with property (T) acts ergodically on G/Γ . Let μ be a probability measure on Λ admissible with respect to some Kazhdan set, and let X_n be the random variable representing the n -th step of the random walk defined by μ .

(i) *If $\sum_n \nu(\Omega_n)$ is finite then for almost every $x \in G/\Gamma$*

$$\sum_{n \in \mathbb{N}} \mathbb{P}(X_n(x) \in \Omega_n) < \infty.$$

(ii) *If $\sum_n \nu(\Omega_n)$ is infinite then for every $\varepsilon > 0$ and for almost every $x \in G/\Gamma$,*

$$(3) \quad \sum_{n \leq N} \mathbb{P}(X_n(x) \in \Omega_n) = S_N + O(S_N^\varepsilon),$$

where $S_N = \sum_{n \leq N} \nu(\Omega_n)$. In particular, $\mathbb{P}(X_n(x) \in \Omega_n) > 0$ for infinitely many $n \in \mathbb{N}$.

Moreover, when Λ is endowed with a word metric dist_Λ corresponding to an arbitrary compact generating, in the theorem above one may obtain an estimate similar to the one in (3) for the smaller probabilities

$$\mathbb{P}(X_n(x) \in \Omega_n, \text{dist}_\Lambda(X_n, e) \geq an),$$

where $a > 0$ is a constant depending on the choice of the word metric and of μ (see Corollary 6.8).

These results apply for instance when G is a semisimple group, Γ a lattice in G and Λ an infinite subgroup of G , or when G/Γ is the n -dimensional torus and Λ is a subgroup of $SL_n(\mathbb{Z})$.

Obstructions to the coarse Baum-Connes conjecture. The final application we present concerns obstructions to the coarse Baum-Connes conjecture. In [26, 27] it was shown that the coarse Baum-Connes conjecture fails for coarse disjoint unions of expander graphs arising from an exact group with property (T). The reason is the existence of a certain Kazhdan-type projection, a non-compact ghost projection, which is a limit of finite propagation operators. Until now such ghost projections were constructed only for expanders as above. Willett and Yu in formulated the following

Problem 1.6 (Problem 5.4, [64]). *Find other geometric examples of ghost projections.*

Here we provide an answer by constructing non-compact ghost projections for warped cones [55].

Let G be a finitely generated group acting ergodically by probability preserving Lipschitz homeomorphisms on a compact metric probability measure space (M, dist, m) . Assume that the measure m is upper uniform, i.e. it is distributed uniformly over M with respect to the metric, see Definition 7.1. Denote by $\mathcal{O} = \mathcal{O}_G(M)$ the warped cone associated to the action of G on M , as defined in Section 7.a (see also [55]).

Theorem 1.7 (see Theorem 7.6). *If the action of G on (M, m) has a spectral gap then there exists a non-compact ghost projection $\mathfrak{G} \in \mathcal{B}(L_2(\mathcal{O}))$ which is a limit of finite propagation operators.*

We also conjecture that such warped cones with non-compact ghost projections as provided by Theorem 1.7 yield a new class of counterexamples to the coarse Baum-Connes conjecture.

Acknowledgments. The second author would like to thank the Mathematical Institute at the University of Oxford for its hospitality during a 4-month stay which made this work possible. Both authors thank Mikael de la Salle, Adam Skalski, Alain Valette, Rufus Willett and Guoliang Yu for valuable comments.

CONTENTS

1. Introduction	1
2. Preliminaries	8
3. Random walks, projections and spectral gaps	11
4. Kazhdan projections in Banach algebras	21
5. Expanders and property (τ) for Banach spaces	25
6. Ergodic theorems	26
7. Ghost projections for warped cones	32
References	38

2. PRELIMINARIES

2.a. Uniform convexity. Let $(E, \|\cdot\|_E)$ be a reflexive Banach space, $\mathcal{B}(E)$ the algebra of bounded linear operators on E and $\mathcal{I}(E)$ the group of linear isometries of E . The *modulus of convexity of E* is the function $\delta_E : [0, 2] \rightarrow [0, 1]$ defined by

$$\delta_E(t) = \inf \left\{ 1 - \left\| \frac{v+w}{2} \right\| ; \|v\| = \|w\| = 1, \|v-w\| \geq t \right\}.$$

The Banach space E is said to be *uniformly convex* if $\delta_E(t) > 0$ for every $t > 0$.

A family $\mathcal{E} = \{E_i\}_{i \in I}$ of Banach spaces is *uniformly convex* if $\delta_{\mathcal{E}}(t) = \inf_{i \in I} \delta_{E_i}(t) > 0$ for every $t > 0$. The function $\delta_{\mathcal{E}}$ is called the *modulus of convexity of the family \mathcal{E}* .

2.b. Admissible measures. Compactly supported probability measures on topological groups and the corresponding random walks are central objects in our arguments. We introduce some notation and several standing assumptions on such measures.

Consider G a locally compact group, endowed with a (left invariant) Haar measure η . For any function $f : G \rightarrow \mathbb{C}$ we denote $\gamma \cdot f(g) = f(\gamma^{-1}g)$, $\gamma, g \in G$.

We consider two particular cases, before introducing the notion of admissible measure in full generality. Let Q be a compact subset of G .

Continuous admissible measures. Let $\alpha, \beta : G \rightarrow [0, \infty)$ be continuous functions with compact support satisfying

$$\int \alpha d\eta = 1 \quad \text{and} \quad \beta(g) \geq s \cdot \alpha(g), \quad \forall s \in Q, g \in G.$$

Another continuous function, whose compact support contains Q , can then be defined by the formula

$$(4) \quad \rho = \frac{\alpha + \beta}{M(\alpha, \beta)}, \quad \text{where} \quad M(\alpha, \beta) = \int_G (\alpha + \beta) d\eta.$$

We call a decomposition as in (4) an (α, β) -decomposition of ρ .

The function ρ gives rise to a probability measure μ on G defined by setting $d\mu = \rho d\eta$.

Discrete admissible measures. Now consider functions with finite support $\alpha, \beta : G \rightarrow [0, \infty)$. With the above conditions formulated for such α and β , we define ρ as in (4), where $M(\alpha, \beta) = \sum_{g \in \text{supp } \alpha \cup \text{supp } \beta} [\alpha(g) + \beta(g)]$, and the formula (4) is again called an (α, β) -decomposition of ρ .

As ρ has finite support and $\sum_{g \in \text{supp } \rho} \rho(g) = 1$, it gives rise to a purely atomic probability measure on G .

Definition 2.1. A measure μ_c (respectively μ_d) will be called a continuous admissible measure with respect to Q (respectively, discrete admissible measure with respect to Q) if it is defined by a continuous density ρ (respectively, a finitely supported function ρ) admitting an (α, β) -decomposition.

A probability measure μ on G will be called admissible with respect to Q if there exists $t \in [0, 1]$ such that $\mu = t\mu_c + (1-t)\mu_d$, where μ_c and μ_d are, respectively, continuous and discrete admissible measures with respect to Q .

The normalizing factor of the function ρ and its associated continuous or discrete measure μ is the infimum of $M(\alpha, \beta)$, taken over all (α, β) -decompositions of ρ , with Q fixed. This factor will be denoted either M_ρ or M_μ , depending on the object referred to.

The normalizing factor of an admissible measure $\mu = t\mu_c + (1-t)\mu_d$ is the number

$$M_\mu = \left(\frac{t}{M_{\mu_c}} + \frac{1-t}{M_{\mu_d}} \right)^{-1}.$$

Admissible measures always exist on a locally compact group. We expect that the arguments presented in this paper would, with some modifications, work for a larger class of measures. However, the above setting allows to identify a continuous admissible measure μ naturally, via the density ρ , with an element of the group algebra $C_c(G)$ (respectively the group ring $\mathbb{C}G$, in the discrete case), which is crucial for further applications.

The set Q will usually be a Kazhdan set (see Definition 2.3). Since such sets can be finite even for Lie groups (see Section 3.g), it is useful to work with measures having an atomic part even in the Lie group setting. When μ is continuous, hence entirely defined by a density ρ with respect to the

Haar measure η , we sporadically replace μ by ρ in the whole notation and terminology.

2.c. Groups and representations. Let G be a locally compact group. An isometric representation $\pi : G \rightarrow \mathcal{B}(E)$ of G on a Banach space E is said to be *continuous* if it is continuous with respect to the strong operator topology. Equivalently, every orbit map is continuous, see [2, Lemma 2.4]. Throughout the article we restrict our attention to representations that are continuous in the above sense, without mentioning this further.

Consider the subspace of E consisting of vectors invariant under π ,

$$E^\pi = \{v \in E : \pi_g v = v \text{ for every } g \in G\}.$$

The dual space E^* is naturally equipped with a contragradient representation $\bar{\pi} : G \rightarrow \mathcal{B}(E^*)$, defined by the formula $\bar{\pi}_g = \pi_{g^{-1}}^*$. Note that $\bar{\pi}$ is isometric if π is, but not necessarily continuous. If E is reflexive we define a subspace $E_\pi = \text{Ann}((E^*)^{\bar{\pi}})$, where Ann denotes the annihilator: the set of all functionals in $E = E^{**}$ that vanish identically on $(E^*)^{\bar{\pi}}$. Both E^π and E_π are π -invariant closed subspaces of E .

Definition 2.2. A representation $\pi : G \rightarrow \mathcal{B}(E)$ is *complemented* if

$$(5) \quad E = E^\pi \oplus E_\pi.$$

A family of complemented representations is called a *complemented family*.

Examples of complemented representations include isometric representations on reflexive Banach spaces [4] (in particular, on uniformly convex Banach spaces [2, Section 2.c]), and representations of small exponential growth of certain Lie groups on Banach spaces with non-trivial Rademacher type [35].

A representation $\pi : G \rightarrow \mathcal{B}(E)$ is *uniformly bounded* if $\|\pi\| = \sup_{g \in G} \|\pi_g\|_{\mathcal{B}(E)} < \infty$. For any such representation π , a new norm can be defined on E , equivalent to the initial one, by the formula

$$(6) \quad \|v\|_\pi = \sup_{g \in G} \|\pi_g v\|.$$

As observed in [2, Proposition 2.3], the modulus of convexity of the norm $\|\cdot\|_\pi$ satisfies $\delta_{\|\cdot\|_\pi}(t) \geq \delta_{\|\cdot\|}(t\|\pi\|^{-1})$ for every $t > 0$.

2.d. Spectral gaps and uniform property (TE). Throughout the section, G is a locally compact group and E a Banach space.

A representation π of G on E has *almost invariant vectors* if for every $\varepsilon > 0$ and every compact subset S in G there exists $v \in E$, $\|v\| = 1$, such that $\sup_{s \in S} \|v - \pi_s v\| \leq \varepsilon$.

Definition 2.3. (i) A complemented representation $\pi : G \rightarrow \mathcal{B}(E)$ has a *spectral gap* if the restriction of π to E_π does not have almost invariant vectors, i.e. if there exists a constant $\kappa > 0$ and a compact subset

Q in G such that for every $v \in E_\pi$

$$\sup_{s \in Q} \|v - \pi_s v\| \geq c \|v\|.$$

Any such pair (Q, κ) is called a Kazhdan pair for π , κ is called a Kazhdan constant, and Q a Kazhdan set.

- (ii) We say that a complemented family \mathcal{F} of representations on a family of Banach spaces has a spectral gap if there exists Q and $\kappa > 0$ as above such that (Q, κ) is a Kazhdan pair for every $\pi \in \mathcal{F}$. We call (Q, κ) a Kazhdan pair for \mathcal{F} .

If \mathcal{F} is a family of representations closed under direct sums then the existence of a spectral gap for each π in \mathcal{F} is equivalent to the existence of a spectral gap for the entire family.

In the particular case when \mathcal{F} is composed of all the unitary representations of a group G , the Kazhdan constant in the sense of the above definition is the classical Kazhdan constant associated to a Kazhdan set. In that case, every generating set of G is a Kazhdan set, but the converse is true only for sets with non-empty interior. See Section 3.g.

The following definition introduces versions of Kazhdan property (T) for Banach spaces.

Definition 2.4. Let \mathcal{E} be a family of Banach spaces and let $E \in \mathcal{E}$.

- (i) G has property (TE) if each isometric representation π of G on E has a spectral gap [2]. More generally, G has property (TE) if every isometric representation of G on any $E \in \mathcal{E}$ has a spectral gap.
- (ii) G has property (TE) uniformly if the family of all isometric representations of G on E has a spectral gap. More generally, G has property (TE) uniformly if the family of all isometric representations of G on all Banach spaces $E \in \mathcal{E}$ has a spectral gap.

It was proved in [2] that if G has property (T) then it has property (TE) for $E = L_p(X, \nu)$, $1 < p < \infty$. It follows that for $E = L_p(0, 1), \ell_p(\mathbb{N})$, for a fixed $1 < p < \infty$, property (TE) holds uniformly. On the other hand, for such G and the class $\mathcal{E} = \{L_p[0, 1], \ell_p(\mathbb{N}) : p \in (1, \infty)\}$, G has property (TE) but not uniformly.

In [5] property $(T\ell_p)$ was studied systematically. It yields a larger class than that of groups with property (T), containing for instance irreducible lattices in products of locally compact second countable groups, one with property (T) and the other with no non-trivial finite dimensional unitary representation.

3. RANDOM WALKS, PROJECTIONS AND SPECTRAL GAPS

3.a. The Markov operator A_π^μ and its properties. Let $\pi : G \rightarrow \mathcal{B}(E)$ be a uniformly bounded representation of G on a Banach space E , and let μ be a

probability measure on G . The operator $A_\pi^\mu : E \rightarrow E$, defined by the Bochner integral

$$A_\pi^\mu v = \int_G \pi_g v d\mu(g)$$

is called the *Markov operator* associated to the random walk on G determined by μ . By standard properties of the Bochner integral, we have that A_π^μ is a bounded operator as well.

The operator A_π^μ can also be defined for bounded representations that are not uniformly bounded, provided that the support of μ is compact.

General properties of A_π^μ . In the following lemma we collect several standard properties that we will need later, of the operator A_π^μ for an isometric representation π . Denote by $\bar{\mu}$ the measure obtained from μ by pre-composing with the map $s \mapsto s^{-1}$ in G and post-composing with the conjugation in \mathbb{C} .

Lemma 3.1. *Let μ be a probability measure on G . Then*

- (i) $(A_\pi^\mu)^* = A_\pi^{\bar{\mu}}$,
- (ii) $A_\pi^{\mu * \nu} = A_\pi^\mu A_\pi^\nu$,
- (iii) $\pi_g A_\pi^\mu = A_\pi^{g \cdot \mu}$ for every $g \in G$,
- (iv) $A_\pi^\mu = I$ on E^π ,
- (v) $A_\pi^\mu(E_\pi) \subseteq E_\pi$.

Properties of A_π^μ with respect to a lattice. Consider now a locally compact group G with a finitely generated lattice Γ , i.e. a finitely generated subgroup Γ such that G/Γ has a finite G -invariant measure induced by the Haar measure.

Let π be a continuous isometric representation of G on a reflexive Banach space E . Denote by $\pi|_\Gamma$ the restriction of π to the lattice Γ . The inclusion of Γ into G gives rise to two decompositions,

$$(7) \quad E = E^\pi \oplus E_\pi = E^{\pi|_\Gamma} \oplus E_{\pi|_\Gamma}.$$

Since $\overline{\pi|_\Gamma} = \overline{\pi|_\Gamma}$, the subspaces above satisfy $E^\pi \subseteq E^{\pi|_\Gamma}$ and $E_{\pi|_\Gamma} \subseteq E_\pi$.

Lemma 3.2. *There is a direct sum decomposition $E_\pi = E_{\pi|_\Gamma} \oplus (E_\pi \cap E^{\pi|_\Gamma})$.*

Proof. Let $v \in E_\pi$. Then $v = w + z$, where $w \in E_{\pi|_\Gamma}$ and $z \in E^{\pi|_\Gamma}$. Thus $z = v - w \in E_\pi$ by (7). \square

The above decomposition is preserved by $\pi|_\Gamma$ but in general not preserved by π .

Now choose a fundamental domain Δ for Γ in G and renormalize the Haar measure η on G so that $\eta(\Delta) = 1$. For the purposes of the next statement denote by A_π^Δ the Markov operator associated to the measure determined by the (possibly discontinuous) characteristic function of Δ as a density function. The next proposition shows that the restriction of A_π^Δ to E_π is concentrated on $E_{\pi|_\Gamma}$.

Proposition 3.3. *Let $v \in E_\pi \cap E^{\pi|\Gamma}$. Then $A_\pi^\Delta v = 0$.*

In particular, given $v \in E_\pi$ we have $A_\pi^\Delta v = A_\pi^\Delta w$, where $w \in E_{\pi|\Gamma}$ is as in the previous lemma.

Proof. Observe that $A_\pi^\Delta v \in E^\pi$. Indeed, for any $h \in G$,

$$\begin{aligned} \pi_h A_\pi^\Delta v &= \int_\Delta \pi_{hg} v d\eta(g) = \int_{h\Delta} \pi_g v d\eta(g) \\ &= \sum_{\gamma \in \Gamma} \int_{h\Delta \cap \Delta\gamma} \pi_g v d\eta(g) = \sum_{\gamma \in \Gamma} \int_{h\Delta\gamma^{-1} \cap \Delta} \pi_{g\gamma} v d\eta(g) \end{aligned}$$

where in the last equality the change of variable $g \mapsto g\gamma^{-1}$ and the unimodularity of G (due to the existence of a lattice) were used.

The above and the fact that v is fixed by Γ imply that

$$\pi_h A_\pi^\Delta v = \sum_{\gamma \in \Gamma} \int_{h\Delta\gamma^{-1} \cap \Delta} \pi_g v d\eta(g) = \int_\Delta \pi_g v d\eta(g) = A_\pi^\Delta v.$$

Since $A_\pi^\Delta v \in E_\pi \cap E^\pi$, the assertion follows. \square

3.b. Proof of Theorem 1.1. Our central result establishes a connection between Kazhdan constants, convergence of iterated Markov operators, and projections onto the subspace of invariant vectors.

Theorem 1.1. *Let G be a locally compact group, and \mathcal{F} a family of isometric representations of G on a uniformly convex family \mathcal{E} of Banach spaces. The following conditions are equivalent:*

- (i) *the family \mathcal{F} has a spectral gap;*
- (ii) *there exists a compactly supported probability measure μ on G and $\lambda < 1$ such that for every isometric representation $\pi \in \mathcal{F}$ of G on $E \in \mathcal{E}$ we have $\|A_\pi^\mu|_{E^\pi}\| < \lambda$;*
- (iii) *there exists a compactly supported probability measure μ on G and a number $\mathfrak{S} < \infty$ such that for every $\pi \in \mathcal{F}$ the iterated Markov operators $(A_\pi^\mu)^k$ converge with speed summable to at most \mathfrak{S} to the projection \mathcal{P}_π onto E^π along E_π .*

Detailed statements and proofs of the implications composing Theorem 1.1 appear in the next three sections: (i) \implies (ii) in section 3.c; (ii) \implies (iii) in section 3.d; and (iii) \implies (i) in section 3.e.

Note that even though the above theorem is stated for isometric representations, it is automatically true for classes of uniformly bounded representations with a common upper bound on norms. This follows from the renormings associated to uniformly bounded representations (6).

We also explain in section 3.f how the above can be extended to a more general setting of a family of groups and a family of their isometric representations on a uniformly convex family of Banach spaces.

3.c. From Kazhdan pairs to contracting Markov operators. In this section, given a Kazhdan pair, we provide a construction of a Markov operator with an effective estimate on the norm on the subspace E_π . The proof we provide relies on uniform convexity.

Theorem 3.4. *Let G be a locally compact group and \mathcal{F} a family of isometric representations of G on a uniformly convex family \mathcal{E} of Banach spaces. Assume that \mathcal{F} has a spectral gap and let (Q, κ) be a Kazhdan pair for \mathcal{F} .*

For every Q -admissible measure μ on G , and for every isometric representation $\pi \in \mathcal{F}$ we have

$$(8) \quad \|A_\pi^\mu|_{E_\pi}\| \leq 1 - \frac{2}{M_\mu} \delta_{\mathcal{E}}(\kappa).$$

Proof. First assume that μ is a continuous admissible probability measure on G . Let π be an isometric representation of G on E with a spectral gap. Since μ is admissible we can choose an (α, β) -decomposition for the density ρ defining μ , as in Section 2.b. We will use α as the upper subscript in reference to the Markov operator associated to the measure determined by the density α .

Let $v \in E_\pi$ be a unit vector. By Lemma 3.1, $A_\pi^\alpha v \in E_\pi$. Fix $s \in Q$ such that

$$\|A_\pi^\alpha v - \pi_s A_\pi^\alpha v\| \geq \kappa \|A_\pi^\alpha v\|.$$

We have

$$(9) \quad A_\pi^\mu v = \left(A_\pi^\mu v - \left(\frac{A_\pi^\alpha v + A_\pi^{s \cdot \alpha} v}{M(\alpha, \beta)} \right) \right) + \left(\frac{A_\pi^\alpha v + \pi_s A_\pi^\alpha v}{M(\alpha, \beta)} \right).$$

We estimate the norm of the first summand in (9) as follows.

$$\begin{aligned} \left\| A_\pi^\mu v - \frac{1}{M(\alpha, \beta)} (A_\pi^\alpha v + A_\pi^{s \cdot \alpha} v) \right\| &= \left\| \int_G \pi_g v \left(\rho - \frac{\alpha + s \cdot \alpha}{M(\alpha, \beta)} \right) d\eta \right\| \\ &\leq \int_G \|\pi_g v\| \left| \rho - \frac{\alpha + s \cdot \alpha}{M(\alpha, \beta)} \right| d\eta \\ &= \int_G \rho - \frac{\alpha + s \cdot \alpha}{M(\alpha, \beta)} d\eta \\ &= 1 - \frac{2}{M(\alpha, \beta)}, \end{aligned}$$

where in the last but one equality we used the fact that $\rho \geq \frac{\alpha + s \cdot \alpha}{M(\alpha, \beta)}$, which follows from the properties of the (α, β) -decomposition.

By the uniform convexity of E , the norm of the second summand in (9) is bounded as follows

$$\begin{aligned} \frac{1}{M(\alpha, \beta)} \|A_\pi^\alpha v + \pi_s A_\pi^\alpha v\| &= \frac{2 \|A_\pi^\alpha v\|}{M(\alpha, \beta)} \left\| \frac{A_\pi^\alpha v + \pi_s A_\pi^\alpha v}{2 \|A_\pi^\alpha v\|} \right\| \\ &\leq \frac{2}{M(\alpha, \beta)} (1 - \delta_E(\kappa)). \end{aligned}$$

The two inequalities together give

$$\begin{aligned}\|A_\pi^\mu v\| &\leq 1 - \frac{2}{M(\alpha, \beta)} + \frac{2}{M(\alpha, \beta)}(1 - \delta_E(\kappa)) \\ &\leq 1 - \frac{2}{M(\alpha, \beta)}\delta_E(\kappa).\end{aligned}$$

Passing to the infimum over all (α, β) -decompositions of ρ gives the estimate

$$\|A_\pi^\mu v\| \leq 1 - \frac{2}{M_\mu}\delta_E(\kappa).$$

The same calculation as above gives the same estimate when μ is a discrete admissible measure. Given any admissible measure $\mu = t\mu_c + (1-t)\mu_d$, $t \in [0, 1]$, for μ_c and μ_d admissible continuous and admissible discrete probability measures, respectively, we have

$$\|A_\pi^\mu\| \leq t\|A_\pi^{\mu_c}\| + (1-t)\|A_\pi^{\mu_d}\|.$$

This yields the required conclusion. \square

In the case of a finite Kazhdan set, we obtain the following.

Corollary 3.5. *Let G be a locally compact group and let \mathcal{F} be a family of isometric representations of G on a uniformly convex family \mathcal{E} of Banach spaces such that $\kappa > 0$ is a Kazhdan constant for a finite Kazhdan set Q , and let $g \notin Q$. Let μ be the uniform probability measure on $Qg \cup \{g\}$. Then for every isometric representation $\pi \in \mathcal{F}$ we have*

$$\|A_\pi^\mu|_{E_\pi}\| \leq 1 - \frac{2}{\#Q + 1}\delta_{\mathcal{E}}(\kappa).$$

Proof. The uniform measure on $Qg \cup \{g\}$ admits an (α, β) -decomposition with α the Dirac mass at $g \in G$ and β the characteristic function of Qg . This decomposition also gives $M_\mu \leq \#Q + 1$ and the estimate follows. \square

Remark 3.6 (Kesten-type characterization in uniformly convex Banach spaces).

In the same spirit, the well-known Kesten characterization of amenability can be generalized as follows. *Let μ be a probability measure with compact support generating the group G . Then a complemented representation π of G has almost invariant vectors in E_π if and only if $\|A_\pi^\mu|_{E_\pi}\| = 1$.*

Remark 3.7. In some particular cases alternative arguments are available to prove Theorem 3.4. For instance, when $\Gamma \subseteq G$ is a lattice generated by a finite set S , by Proposition 3.3 we have that the norm of the Markov operator A_π^μ needs to be computed on the vectors in $E_{\pi|\Gamma}$ only. Let $v \in E_{\pi|\Gamma}$ and $s \in \Gamma$ be a generator for which the Kazhdan constant $\kappa > 0$ is attained for v . Since s is an element of the lattice Γ , we can choose a sufficiently small neighborhood U of the identity, with positive Haar measure, satisfying $U \cap Us = \emptyset$, and write $\|A_\pi^\mu v\| \leq \|\int_{G \setminus U \cup Us} \pi_g v d\mu\| + \|\int_{Us} \pi_g v d\mu + \int_U \pi_g v d\mu\|$. The first term is bounded from above by $1 - \mu(U) - \mu(Us)$, while the second term can be estimated using the modulus of uniform convexity and bounds

on Radon-Nikodym derivatives of the measure $s * \mu$ with respect to μ . Since in general we can assume these bounds to be universal over all Us , $s \in S$, we can obtain the final claim that $\|A_\pi^\mu\| < 1$.

3.d. From contracting Markov operators to projections. Recall that the *Neumann series of an operator* T is the series $\sum_{n=0}^\infty T^n$. It is convergent if $\|T\| < 1$ and in that case it is the inverse of $I - T$. This allows to give an explicit formula for the projection onto invariant vectors in terms of the Markov operator.

Theorem 3.8. *Let G be a locally compact group and μ a probability measure on G . Let \mathcal{F} be a complemented family of isometric representations of G on a family of Banach spaces \mathcal{E} . If there exists $\lambda < 1$ such that for every representation $\pi \in \mathcal{F}$ on $E \in \mathcal{E}$ we have $\|A_\pi^\mu|_{E_\pi}\| \leq \lambda$ then for every $\pi \in \mathcal{F}$*

(i) *the operator*

$$\mathcal{P}_\pi = I_E - \left(\sum_{n=0}^\infty (A_\pi^\mu)^n \right) (I_E - A_\pi^\mu)$$

is the projection $E \rightarrow E^\pi$ along E_π onto the subspace of invariant vectors of the representation π ;

(ii) *the iterated average operator $(A_\pi^\mu)^k$ converges to \mathcal{P}_π exponentially fast, uniformly over \mathcal{F} ,*

$$\|(A_\pi^\mu)^k - \mathcal{P}_\pi\| \leq \lambda^k.$$

Proof. (i). The operator $I - A_\pi^\mu$ is invertible on E_π and its inverse on E_π is given by the Neumann series

$$(I_{E_\pi} - A_\pi^\mu)^{-1} = \sum_{n=0}^\infty (A_\pi^\mu)^n.$$

With this in mind we proceed to show that \mathcal{P}_π is well-defined. We observe that for every $v \in E$ we have $(I_E - A_\pi^\mu)v \in E_\pi$. Indeed, using the decomposition (5) we can write $v = z + w$ uniquely, where $z \in E^\pi$ and $w \in E_\pi$. We have

$$(I_E - A_\pi^\mu)v = w - A_\pi^\mu w \in E_\pi,$$

since $(I_E - A_\pi^\mu)z = 0$, by Lemma 5. Since $(I - A_\pi^\mu)v \in E_\pi$ and the Neumann series of A_π^μ converges on E_π , we see that \mathcal{P}_π is well-defined and bounded.

Observe also that since $(I - A_\pi^\mu)z = 0$ for $z \in E^\pi$, we have $\mathcal{P}_\pi z = z$. On the other hand, if $w \in E_\pi$ then

$$\left(\sum_{n=0}^\infty (A_\pi^\mu)^n \right) (I_E - A_\pi^\mu)w = w,$$

and, consequently, $\mathcal{P}_\pi w = 0$. Therefore, given any vector $v = z + w$ as above we have $\mathcal{P}_\pi(z + w) = z$.

(ii). Observe \mathcal{P}_π is a norm limit of operators $\mathcal{P}_{\pi,k}$, defined by truncating the Neumann series to its k -partial sum, and $\mathcal{P}_{\pi,k} = (A_\pi^\mu)^{k+1}$. We prove the

inequality by induction on k . For $k = 1$ we can write

$$\|A_\pi^\mu - \mathcal{P}_\pi\| = \sup \{\|A_\pi^\mu w\| : w \in E_\pi, z \in E^\pi, \|z + w\| = 1\} \leq \lambda.$$

Assume that the inequality is proven for k . Since $A_\pi^\mu \circ \mathcal{P}_\pi = \mathcal{P}_\pi$ and since the image of $(A_\pi^\mu)^k - \mathcal{P}_\pi$ is in E_π , we can write

$$\|(A_\pi^\mu)^{k+1} - \mathcal{P}_\pi\| = \|(A_\pi^\mu)^{k+1} - A_\pi^\mu \circ \mathcal{P}_\pi\| \leq \lambda \|(A_\pi^\mu)^k - \mathcal{P}_\pi\| \leq \lambda^{k+1}.$$

□

In the particular case of finite Kazhdan sets, one can replace the iteration of an average operator by products of average operators. Indeed, let G be a locally compact group and let \mathcal{F} be a family of isometric representations of G on a uniformly convex family of Banach spaces, \mathcal{F} admitting a Kazhdan pair (X, κ) with $X = \{x_1, \dots, x_N\}$. Let S_n be a sequence of finite sets constructed in one of the following manners

- (a) $S_n = X \cup Y_n$, where $Y_n = \{y_1, \dots, y_M\}$ is an arbitrary subset of M elements in G , where M is fixed;
- (b) $S_n = X_n \cup Y_n$, where $Y_n = \{y_1, \dots, y_N\}$ and $X_n = \{y_i x_i y_i^{-1} : 1 \leq i \leq N\}$.

Let μ_n be the atomic uniform measure on the set S_n .

Corollary 3.9. *For the sequence of measures μ_n constructed above and for every representation $\pi \in \mathcal{F}$ we have*

$$\|A_\pi^{\mu_1} A_\pi^{\mu_2} \dots A_\pi^{\mu_n} - \mathcal{P}_\pi\| \leq \left(1 - \frac{2}{N} \delta_E(\kappa/3)\right)^n.$$

Proof. For $S_n = X \cup Y_n$ the Kazhdan constant is at least κ . Let now $S_n = X_n \cup Y_n$. For every $v \in E_\pi$, $\|v\| = 1$, there exists $x \in X$ such that $\|\pi_x v - v\| \geq \kappa$. The triangle inequality allows to write, for the corresponding element $y \in Y_n$ that

$$\kappa \leq \|\pi_y v - v\| + \|\pi_{y^{-1}} v - v\| + \|\pi_{yxy^{-1}} v - v\|.$$

It follows that at least one of the terms in the sum above is larger than $\frac{\kappa}{3}$. Therefore in this case the Kazhdan constant is at least $\frac{\kappa}{3}$. This, Corollary 3.5 and an easy induction on n yields the inequality. □

Remark 3.10 (Constructing almost invariant vectors). The convergence with controlled speed of iterated Markov operators to the projection onto the space of fixed vectors allows to produce almost invariant vectors with an arbitrarily small degree of almost invariance. In particular, it allows to design a non-deterministic algorithm which, given a vector, never stops if the vector has no component in the subspace of fixed vectors, while if it stops it produces an almost invariant unit vector with the desired degree of almost invariance (equivalently, an approximation with the desired order of error of a fixed unit vector).

The estimates on the norm of Markov operators also allow to compute explicit mixing times, which would again be uniform for all vectors in E_π .

Note that this can be achieved not only for finitely generated groups, but also for topological groups that admit finite Kazhdan sets (see section 3.g).

3.e. From projections to spectral gaps. Finally, we show that a summable convergence of $(A_\pi^\mu)^k$ to the projection onto the subspace of fixed points implies the existence of a spectral gap.

Theorem 3.11. *Let μ be a compactly supported probability measure on G and \mathcal{F} be a complemented family of isometric representations of G on a family \mathcal{E} of Banach spaces. Assume that there exists a number $\mathfrak{S} < \infty$ such that for every representation $\pi \in \mathcal{F}$ on $E \in \mathcal{E}$ the iterated Markov operators $(A_\pi^\mu)^k$ converge to a bounded operator \mathcal{P} and*

- (i) $E_\pi \subseteq \ker \mathcal{P}$,
- (ii) *the convergence has speed summable to at most \mathfrak{S} , i.e. there exists a sequence of positive numbers a_k such that the series $\sum_{k \in \mathbb{N}} a_k$ converges to a finite number $\leq \mathfrak{S}$ and*

$$(10) \quad \left\| (A_\pi^\mu)^k - \mathcal{P} \right\| \leq a_k.$$

Then the family \mathcal{F} has a spectral gap and $(\text{supp } \mu, \frac{1}{1+\mathfrak{S}})$ is Kazhdan pair.

Proof. Denote $Q = \text{supp } \mu$. Let $\pi \in \mathcal{F}$ be an isometric representation of G on $E \in \mathcal{E}$ and $v \in E_\pi$ be an arbitrary unit vector. Let $\sigma_v = \sup_{g \in Q} \|\pi_g v - v\|$. Then

$$(11) \quad \|A_\pi^\mu v - v\| \leq \sigma_v.$$

For every positive integer k we can then write

$$\begin{aligned} \left\| (A_\pi^\mu)^k v - v \right\| &\leq \|A_\pi^\mu v - v\| + \sum_{i=1}^{k-1} \left\| (A_\pi^\mu)^{i+1} v - (A_\pi^\mu)^i v \right\| \\ &\leq \|A_\pi^\mu v - v\| \left(1 + \sum_{i=1}^{k-1} \left\| (A_\pi^\mu)^i \right\|_{E_\pi} \right) \\ &\leq \sigma_v \left(1 + \sum_{i=1}^{k-1} a_k \right). \end{aligned}$$

Then

$$1 \leq \left\| (A_\pi^\mu)^k v - v \right\| + \left\| (A_\pi^\mu)^k v \right\| \leq \sigma_v (1 + \mathfrak{S}) + a_k.$$

Since the above is true for every $k \in \mathbb{N}$ we obtain

$$\sigma_v \geq \frac{1}{1 + \mathfrak{S}}.$$

□

The above estimates have several interesting consequences. The first is that as soon as the upper bounds $a_k = \left\| (A_\pi^\mu)^k - \mathcal{P} \right\|$ compose a convergent series, they must be decreasing exponentially, \mathcal{P} must be the projection onto E^π along E_π , and we are in the presence of a spectral gap.

Another consequence of the previous results is that the Kazhdan constant and the norm $\|A_\pi^\mu|_{E_\pi}\|$ are closely related. We will state this result for finite Kazhdan sets and uniform measures, since in this case the formulation is particularly concise. The general case can be deduced in the same manner.

Theorem 3.12. *Let G be a locally compact group and let \mathcal{F} be a family of isometric representations of G on a family \mathcal{E} of uniformly convex Banach spaces. Assume that \mathcal{F} has a spectral gap and let (Q, κ) be a Kazhdan pair for \mathcal{F} , where Q is finite. If μ is the uniform measure on $Qg \cup \{g\}$, for an arbitrary element $g \in G$ then for every representation $\pi \in \mathcal{F}$ we have*

$$1 - \kappa \leq \|A_\pi^\mu|_{E_\pi}\| \leq 1 - \frac{2}{\#Q + 1} \delta_E(\kappa).$$

Proof. The upper bound follows from Theorem 1.1, in particular Corollary 3.5. To prove the lower bound note that for a unit vector $v \in E_\pi$ the inequality (11) yields

$$1 - \sigma_v = \|v\| - \sigma_v \leq \|A_\pi^\mu v\|.$$

Passing to the supremum over $v \in E_\pi$ of norm 1 on both sides we obtain the claim. \square

Remark 3.13. In the particular case when \mathcal{E} is a family of Hilbert spaces the lower bound on the Kazhdan constant in terms of the norm of the Markov operator $\kappa \geq 1 - \|A_\pi^\mu|_{E_\pi}\|$ can be improved to

$$\kappa \geq \sqrt{2} \sqrt{1 - \|A_\pi^\mu|_{E_\pi}\|}.$$

This is obtained using the argument in [59, p. 842].

A problem formulated by Serre and de la Harpe-Valette [13, 25] asks to compute explicit Kazhdan (sets and) constants, for unitary representations on Hilbert spaces. In the case of representations on uniformly convex Banach spaces, Theorem 3.12 implies the following.

Corollary 3.14. *Let G be a locally compact group and let \mathcal{F} be a complemented family of isometric representations of G on Banach spaces for which Q is a finite Kazhdan set. Then for every $\varepsilon > 0$ there exists an integer $m \in \mathbb{N}$ such that $(\overline{Q}^m, 1 - \varepsilon)$ is a Kazhdan pair for \mathcal{F} , where $\overline{Q} = Q \cup \{e\}$.*

Proof. Given μ the uniform probability measure on \overline{Q} and π an arbitrary representation in \mathcal{F} , by Corollary 3.9 we have $\|(A_\pi^\mu)^k|_{E_\pi}\| = \|A_\pi^{\mu^k}|_{E_\pi}\| \leq \lambda^k$ for some $\lambda \in (0, 1)$, where $\mu^k = \mu * \dots * \mu$. Since the support of the probability measure μ^k is $X = \overline{Q}^k$, the argument in the proof of Theorem 3.11, which yields the inequality in Theorem 3.12, implies that the Kazhdan constant for X is at least $1 - \lambda^k$. \square

3.f. Remarks.

Remark 3.15 (Families of groups). Instead of a single group G we can consider a family $\mathcal{G} = \{G_i\}$ of locally compact groups, each with a corresponding subset Q_i and a probability measure μ_i as in section 2.b. Then we can consider a family \mathcal{F} , whose elements are isometric representations $\pi^{(i)}$ of the groups G_i on uniformly convex Banach spaces $E^{(i)}$, each of which belongs to a uniformly convex family \mathcal{E} . If, in this setting, the parameters that appear in

the previous arguments (Kazhdan constants, normalizing factors M_{μ_i} of the measures μ_i , moduli of convexity etc.) have estimates uniform in i , then the proofs of the previous theorems yield the same estimates as in the case of a single group, uniformly in \mathcal{F} . We leave the details to the reader.

Remark 3.16 (Hereditary properties). Property (T) is preserved by interchanging a locally compact group with its lattice and by extensions of Kazhdan groups by Kazhdan groups, see e.g. [6]. A similar fact is true in our case as well, provided certain assumptions are satisfied. First, the lattice has to be p -integrable, or at least cocompact, see [2] for the definition of p -integrable lattices and for further details. In that case p -induction can be used to induce representations of G from representation of Γ . The second requirement is that the class of representations we are considering is closed under such induction. Examples include the class of isometric representations on L_p -spaces for a fixed $p \in (1, \infty)$, or the class of isometric representations on uniformly convex Banach spaces. Since these methods are standard we leave the details to the reader.

3.g. Finite Kazhdan sets. For many applications the existence of finite Kazhdan sets is a considerable asset, as the averages become finite, the random walks discrete, and an algorithmical approach and the use of computer become possible (see for instance Theorem 3.8, Remark 3.10 and Section 6). As it turns out, the existence of such finite sets is granted in many cases. Shalom formulated a property that he called the *strong property (T)* (st.pr. (T)), requiring the existence of a finite Kazhdan set, and proved that many property (T) groups satisfy it. This theme meets another more recent one, which is the existence of a spectral gap of Hecke operators [7, 8].

In [59], Y Shalom proved st.pr. (T) for groups of k -points of a simply connected, semisimple, almost k -simple group of rank at least 2 (where k is a locally compact non-discrete field), with explicit descriptions of finite Kazhdan sets and their corresponding constants. Theorem C in [59] implies that in a semisimple Lie group with finite center, every finite symmetric set not contained in a closed amenable subgroup is a Kazhdan set. In a second paper [60], Shalom proved that any connected Lie group with property (T) that does not have \mathbb{R}/\mathbb{Z} as a quotient has st. pr (T).

In the case of a compact group G , st. pr (T) has a very interesting equivalent. In that case, for the regular representation of G on $L_2(G)$, when the measure μ is supported on a finite symmetric set $\{g_1^{\pm 1}, \dots, g_m^{\pm 1}\} \subset G$, the averaging operator $2mA_\pi^\mu$ is also known as the *Hecke operator* and is sometimes also denoted z_{g_1, \dots, g_m} . This operator is said to have a *spectral gap* if its norm on the space $L_2^0(G)$ of functions orthogonal to the constants is at most $2m - \zeta$.

The following double implication, which essentially amounts to an equivalence between spectral gap and $\{g_1^{\pm 1}, \dots, g_m^{\pm 1}\}$ being a Kazhdan set, then holds:

- (i) if the Hecke operator z_{g_1, \dots, g_m} satisfies $\|z_{g_1, \dots, g_m}\| \leq 2m - \zeta$ on $L_2^0(G)$, where $\zeta > 0$, then $(\{g_1, \dots, g_m\}, \sqrt{\zeta/m})$ is a Kazhdan pair;
- (ii) given a Kazhdan pair (Q, ϵ) , and an arbitrary $g \notin Q$, the Hecke operator $z_{\{g\} \cup Q_g}$ has spectral gap with $\zeta \geq 2m - 2 + \sqrt{4 - \kappa^2}$.

Results of Bourgain and Gamburd [7, 8] then give, *via* the equivalence described above, many explicit finite Kazhdan sets for the groups $SU(d)$, with Kazhdan constants explicitly computable from constants appearing in a certain noncommutative Diophantine property satisfied by the given set.

4. KAZHDAN PROJECTIONS IN BANACH ALGEBRAS

4.a. Group Banach algebras and projections. Let G be a locally compact group and denote by $C_c(G)$ the group algebra of compactly supported continuous functions on G with convolution. Let \mathcal{F} be a family of representations $\pi : G \rightarrow \mathcal{B}(E)$, by bounded operators on Banach spaces E in a given family \mathcal{E} . For the purposes of this section we also assume that \mathcal{F} contains the trivial representation on at least one $E \in \mathcal{E}$.

Assuming that for each $f \in C_c(G)$, $\sup \{\|\pi(f)\|_{\mathcal{B}(E)} : \pi \in \mathcal{F}\} < \infty$, we equip the algebra $C_c(G)$ with the norm $\|f\|_{\mathcal{F}} = \sup_{\pi \in \mathcal{F}} \|\pi(f)\|_{\mathcal{B}(E)}$.

Definition 4.1. *The algebra $C_{\mathcal{F}}(G)$ is the completion of $C_c(G)$ in the norm $\|\cdot\|_{\mathcal{F}}$.*

If $\bar{\pi} \in \mathcal{F}$ whenever $\pi \in \mathcal{F}$ and the class \mathcal{E} is stable under complex conjugation then $C_{\mathcal{F}}(G)$ admits a natural involution.

The classical example of algebra of type $C_{\mathcal{F}}(G)$ is the maximal group C^* -algebra $C_{\max}^*(G)$, corresponding to \mathcal{F} being the family of all unitary representations of G . Other examples include the following algebras.

- The L_p -maximal group algebra, where $p \in (1, \infty)$, denoted by $C_{\max}^p(G)$. This algebra corresponds to the class \mathcal{F} of all isometric representations of G on L_p -spaces.
- Uniformly bounded group algebras, corresponding to the choice of \mathcal{F} as a class $[\mathcal{E}; k]$ composed of all the uniformly bounded representations of G on $E \in \mathcal{E}$ satisfying $\|\pi\| \leq k$, where $k \geq 1$ and \mathcal{E} is a uniformly convex family of Banach spaces.
- Small exponential growth group algebras. Let ℓ be a continuous length function on G , and let \mathcal{E} be a family of Banach spaces closed under duality and complex conjugation. A representation $\pi : G \rightarrow \mathcal{B}(E)$ on $E \in \mathcal{E}$ is said to have (s, c) -small exponential growth, for some $s, c > 0$, if $\|\pi_g\| \leq ce^{s\ell(g)}$ for every $g \in G$. We denote the class of all such representations by $\mathcal{L}(\ell, s, c)$, and we call the algebra $C_{\mathcal{L}(\ell, s, c)}(G)$ a small exponential growth algebra.

The algebra $C_{\max}^p(G)$ is an immediate natural generalization of the maximal group C^* -algebra, see [51, 22]. Such algebras are relevant for an L^p -approach to the Novikov and the Baum-Connes conjectures [29, 17].

The algebra $C_{[\mathcal{E};k]}$ and the corresponding notions of property (T) for uniformly bounded representations are related to a conjecture of Y. Shalom that every hyperbolic group has an affine action on a Hilbert space with linear part uniformly bounded, see [48, Problem 14] and [46] for related results. This conjecture has attracted a lot of interest lately.

Definition 4.2. A Kazhdan projection in $C_{\mathcal{F}}(G)$ is a central idempotent $p \in C_{\mathcal{F}}(G)$ such that $\pi(p) = \mathcal{P}_{\pi}$ for every representation $\pi \in \mathcal{F}$.

Kazhdan projections are important already in the setting of unitary representations [18, 28]. Their existence in certain algebras $C_{\mathcal{L}(\ell,s,c)}(G)$ is also particularly significant, as they are used by V. Lafforgue to define strong Banach property (T). The latter property is relevant to the Baum-Connes conjecture, see [37, 52].

Definition 4.3 ([35]). The group G has the strong Banach property (T) for \mathcal{E} , denoted $(T_{\mathcal{E}}^{Ban})$, if for every continuous length function ℓ on G there exists $s > 0$ such that for every $c > 0$ the algebra $C_{\mathcal{L}(\ell,s,c)}(G)$ contains a Kazhdan projection.

In the case of representations with small exponential growth we record the following

Theorem 4.4. Let G be a locally compact group, and \mathcal{E} a class of Banach spaces closed under duality and complex conjugation.

- (i) The following are equivalent:
 - (a) G has the property $(T_{\mathcal{E}}^{Ban})$;
 - (b) for every continuous length function ℓ , there exists $s > 0$ such that for every $c > 0$, $\mathcal{L}(\ell, s, c)$ is complemented, and there exists $\rho \in C_c(G)$ satisfying $\int_G \rho d\eta = 1$, and $\lambda < 1$, such that $\|A_{\pi}^{\rho}|_{E_{\pi}}\| < \lambda$ for every $\pi \in \mathcal{L}(\ell, s, c)$;
 - (c) for every continuous length function ℓ , there exists $s > 0$ such that for every $c > 0$, $\mathcal{L}(\ell, s, c)$ is complemented, and there exists ρ and λ as above such that ρ^n converge exponentially fast to some element $p \in C_{\mathcal{L}(\ell,s,c)}(G)$,

$$\|\rho^n - p\|_{\mathcal{L}(\ell,s,c)} \leq \lambda^n.$$

- (ii) If G has the property $(T_{\mathcal{E}}^{Ban})$ then for the corresponding ℓ and s and an arbitrary $c > 0$, the pair $(\text{supp } \rho, \frac{1-\lambda}{a_{\rho}})$ is a Kazhdan pair for the family $\mathcal{F} = \mathcal{L}(\ell, s, c)$, where $a_{\rho} = \int |\rho(g)| d\eta(g)$.

Remark 4.5. The main difference in comparison to Theorem 1.1, is that the Markov operators are defined by signed measures. It is unclear if ρ can always be chosen to be non-negative in this setting. In all the cases in which Kazhdan projections have been constructed in the small exponential growth algebras, they are in fact limits of positive functions with compact support and of integral 1 [35, 38, 57].

Proof. (ia) \Rightarrow (ib). The fact that $\mathcal{L}(\ell, s, c)$ is complemented follows from the fact that \mathcal{E} is closed under duality. There exist $p_n \in C_c(G)$ of integral 1 converging to p in $C_{\mathcal{L}(\ell, s, c)}(G)$. Therefore, one can choose $\rho = p_n$ for n large enough.

(ib) \Rightarrow (ic) is proved exactly as the similar implication in Theorem 1.1.

(ic) \Rightarrow (ia) is straightforward.

(ii) Using the equivalence (ia) \Leftrightarrow (ic) and the argument in the proof of 3.11, one obtains the required conclusion. \square

In the case of isometric representations we have the following characterization of the existence of Kazhdan projections.

Theorem 4.6. *Let \mathcal{F} be a family of isometric representations of a locally compact group G on a uniformly convex family \mathcal{E} . There exists a Kazhdan projection in $C_{\mathcal{F}}(G)$ if and only if the family \mathcal{F} has a spectral gap.*

Moreover, p is always a limit of a sequence of positive probability measures.

Proof. Let Q be a Kazhdan set for \mathcal{F} , and let $\rho \in C_c(G)$ be a density function of a continuous admissible measure μ with respect to Q . It suffices to show that $\{\rho^k\}_{k \in \mathbb{N}}$ is a Cauchy sequence with respect to the norm $\|\cdot\|_{\mathcal{F}}$.

For any representation $\pi \in \mathcal{F}$ and $m < n$ we again have that the image of $I_E - (A_\pi^\mu)^n$ is in E_π and

$$\begin{aligned} \|\pi(\rho^m) - \pi(\rho^n)\|_{\mathcal{B}(E)} &= \|(A_\pi^\mu)^m (I_E - (A_\pi^\mu)^{n-m})\| \\ &\leq \|A_\pi^\mu|_{E_\pi}\|^m \|I_E - (A_\pi^\mu)^{n-m}\| \\ &\leq 2 \|A_\pi^\mu|_{E_\pi}\|^m \end{aligned}$$

Since the last term satisfies a uniform estimate $\|A_\pi^\mu|_{E_\pi}\|^m \leq \lambda^m$ for some $\lambda \in (0, 1)$ and every $\pi \in \mathcal{F}$, the sequence ρ^k is indeed a Cauchy sequence in the \mathcal{F} norm, as claimed, $\|\rho^m - \rho^n\|_{\mathcal{F}} \leq \lambda^m$. There exists a limit, denoted p , which is the required Kazhdan projection. Indeed, a similar estimate as above shows that $\|\rho^{2m} - \rho^m\|_{\mathcal{F}} \leq 2\|\rho^m\|_{\mathcal{F}} \rightarrow 0$, hence p is an idempotent. Finally, since $\pi(p)$ commutes with $\pi(g) = \pi_g$ for every $g \in G$ and every π in \mathcal{F} , we see that p is central.

The converse follows from Theorem 4.4, (ii). It was also proved in [35]. \square

As an application, we have the following generalization of the existence of Kazhdan projections to the class of L_p -maximal group algebras.

Corollary 4.7. *If \mathcal{F} is the family of all isometric representation of G on E then then there exists a Kazhdan projection in $C_{\mathcal{F}}(G)$ if and only if G has uniform property (TE).*

In particular, if G has Kazhdan's property (T) then for every $1 < p < \infty$ there exists a Kazhdan projection in the L_p -maximal group algebra $C_{\max}^p(G)$.

All the statements in Theorems 4.4 and 4.6 are also true, with the appropriate changes, in the more general case of uniformly bounded representations.

In the case of finitely generated groups it was shown in [34] that property (TE) is equivalent to the fact that for every isometric representation π of G on E the projection onto E^π is in the closure of $\{\pi(f) : \text{supp } f < \infty\} \subseteq \mathcal{B}(E)$.

4.b. Relations between versions of property (T) for Banach spaces.

The above results have another important consequence: for uniformly convex Banach spaces they put on the common ground the reinforced Banach property $(T_{\mathcal{E}}^{Ban})$, introduced by V. Lafforgue in [35] and the properties (TE) and FE , introduced in [21, 2]. We refer to [47] for a survey of these properties. The question of such a relation was considered by several experts. In particular, it appeared as Question 1.10 in [14].

To make a direct comparison consider a uniformly convex family \mathcal{E} of Banach spaces, closed under duality and complex conjugation, and the class \mathcal{F} of all isometric representations of G on all $E \in \mathcal{E}$.

Recall that G has property FE for a Banach space E if every continuous affine isometric action of G on E has a fixed point [21, 2]. Equivalently, $H^1(G, \pi) = 0$ for every isometric representation π of G on E . We will say that G has property $F\mathcal{E}$ for a family of Banach spaces \mathcal{E} if G has property FE for every $E \in \mathcal{E}$.

It follows from the results of this paper that we have the implications:

$$\begin{array}{ccccc}
 \text{Lafforgue's } (T_{\mathcal{E}}^{Ban}) & \xrightleftharpoons{\text{[36]}} & F\mathcal{E} & \xrightleftharpoons{\text{[2]}} & (T\mathcal{E}) \\
 \Downarrow \text{(by def.)} & & & & \Uparrow \text{(by def.)} \\
 \text{Kazhdan projection in } C_{\mathcal{F}}(G) & \xrightleftharpoons{\text{Theorem 4.6}} & & & \text{uniform } (T\mathcal{E})
 \end{array}$$

As mentioned earlier, uniformity of property $(T\mathcal{E})$ is automatic if the class \mathcal{E} is closed under taking infinite direct sums and then the vertical arrow on the right is an equivalence. We also remark that Lafforgue's proof of the first implication $(T_{\mathcal{E}}^{Ban}) \implies F\mathcal{E}$ does in fact use linearly growing representations in an essential way [36].

We also observe that the existence of a Kazhdan projection in $C_{\mathcal{F}}(G)$ does not in general imply $F\mathcal{E}$. The reason is the existence of hyperbolic groups with property (T) . Such groups all have (TL_p) for every $1 < p < \infty$, but for $p > 2$ sufficiently large every hyperbolic group admits an unbounded, or even a proper affine isometric action on some L_p -space [10, 65, 44].

Remark 4.8 (Uniform property FE). We take this opportunity to remark that it is possible also to define uniform property FE , as we very briefly describe here. Property FE is the same as vanishing of $H^1(G, \pi)$ for every isometric representation π of G on E . That is, the codifferential $\delta_\pi : E \rightarrow \ker d_\pi$ is onto, where $\ker d_\pi$ is the space of 1-cocycles. This on the other hand is equivalent to the adjoint map $\delta_\pi^* : (\ker d_\pi)^* \rightarrow E^*$ being bounded below, i.e.

$\|\delta^* \varphi\| \geq C_\pi \|\varphi\|$ for every $\varphi \in (\ker d_\pi)^*$ and $C_\pi > 0$. This fact is used extensively in [3, 46] and we refer the reader to those articles for details. We can now define uniform property FE as property FE , where the constant $C_\pi > 0$ can be chosen uniformly for all isometric representations π . We note however that such a uniform choice of C_π is equivalent to having a uniform spectral gap for all isometric representations on E . Thus uniform property FE is simply property FE together with uniform property (TE) .

5. EXPANDERS AND PROPERTY (τ) FOR BANACH SPACES

Let G be a finitely generated group and Q a finite subset of G . Assume that G is residually finite, and consider a sequence $\mathcal{N} = \{N_i\}_{i \in \mathbb{N}}$ of finite index normal subgroups, satisfying $\bigcap_{i \in \mathbb{N}} N_i = \{e\}$. Let $q_i : G \rightarrow G/N_i$ be the quotient map for every $i \in \mathbb{N}$. For $v, u \in G/N_i$ we write $v \sim u$ if v and u are joined by an edge in the Cayley graphs $\text{Cay}(G/N_i, q_i(Q))$.

Given a uniformly convex Banach space E , we denote by $\mathcal{N}(E)$ the family of representations $\pi^{(i)}$ of G on the spaces $\ell_2(G/N_i, E)$ given by $\pi_g^{(i)} f(v) = f(g^{-1}v)$. The projection $E \rightarrow E^{\pi^{(i)}}$ is simply given by the average $M_i f = [G : N_i]^{-1} \sum_{v \in G/N_i} f(v)$.

Property (τ) was defined by A. Lubotzky, see [39, 40]. Here we define a generalization of property (τ) to the setting of uniformly convex Banach spaces.

Definition 5.1. *Let E be a uniformly convex Banach space. A residually finite finitely generated group G has property (τ_E) with respect to \mathcal{N} if the family $\mathcal{N}(E)$ has a spectral gap.*

The above definition naturally generalizes the condition known from Hilbert spaces, that the family of representations $\mathcal{N}(E)$ is separated from the trivial representation in the Fell topology.

Definition 5.2. *A sequence $\{\Gamma_i\}_{i \in \mathbb{N}}$ of graphs is a sequence of E -expanders if it satisfies a Poincaré inequality for E -valued functions uniformly; that is, there exists a constant $\kappa > 0$ such that the Poincaré inequality*

$$\sum_{v \in \Gamma_i} \|f(v) - Mf\|_E^2 \leq \kappa \sum_{v \sim u} \|f(v) - f(u)\|_E^2$$

holds for every $f : \Gamma_i \rightarrow E$ for every $i \in \mathbb{N}$.

We show that property (τ_E) gives the correct generalization of property (τ) to the setting of uniformly convex spaces.

Theorem 1.3. *Let E be a uniformly convex Banach space, G be a finitely generated residually finite group and let $\mathcal{N} = \{N_i\}$ a collection of finite index subgroups with trivial intersection. The following conditions are equivalent:*

- (i) *G has property (τ_E) with respect to $\mathcal{N} = \{N_i\}$ and a symmetric Kazhdan set Q ;*
- (ii) *the Cayley graphs $\text{Cay}(G/N_i, Q)$ form a sequence of E -expanders;*
- (iii) *there exists a Kazhdan projection $p \in C_{\mathcal{N}(E)}(G)$.*

Proof. The equivalence of (i) and (iii) follows from Theorem 4.6. The implication (ii) \implies (i) is clear. To prove the converse we recall an argument from [35]. If E is uniformly convex then the spaces $\ell_2(G/N_i, E)$ are also uniformly convex with the same modulus of uniform convexity. Choose a probability measure μ defined by a density function ρ whose support is the Kazhdan set $Q \subset G$. By assumption on the spectral gap and by Theorem 1.1, the convergence of $\pi^{(i)}(\rho^k)$ to the projection onto the invariant vectors is thus uniform in $i \in \mathbb{N}$. In particular, there exists $k \in \mathbb{N}$ such that for every $i \in \mathbb{N}$, $\|\pi^{(i)}(\rho^k) - M_i\|_{\mathcal{B}(\ell_2(G/N_i))} \leq \frac{1}{2}$. Given an arbitrary $f \in \ell_2(G/N_i, E)$ we have

$$\begin{aligned} \|f - M_i f\| &\leq \|\pi^{(i)}(\rho^k)(f) - M_i f\| + \|f - \pi^{(i)}(\rho^k)f\| \\ &\leq \frac{1}{2} \|f - M_i f\| + \|f - \pi^{(i)}(\rho^k)f\|, \end{aligned}$$

whence

$$(12) \quad \|f - M_i f\| \leq 2 \|f - \pi^{(i)}(\rho^k)f\|.$$

Since $\rho^k \in \mathbb{C}G$ is supported in a ball of radius k for a word metric defined by a set containing Q , we have $\|f - \pi^{(i)}(\rho^k)f\| \leq \#B(e, k)^3 k^3 \sum_{v \sim w} \|f(v) - f(w)\|^2$. Since k is the same regardless of $i \in \mathbb{N}$, the estimate is uniform. \square

Note that in the case when E is a Hilbert space the algebra $C_{\mathcal{N}(E)}(G)$ is a C^* -algebra.

Remark 5.3. We also point out that in the setting of uniformly convex Banach spaces one more characterization of property (τ_E) is true, namely that G has (τ_E) with respect to $\{N_i\}$ if and only if the cohomology $H^1(G, \pi) = 0$, where $\pi = \oplus \pi^{(i)}$. The proof given for Hilbert spaces in [41] can be copied verbatim to the above setting as it uses only the Open Mapping Theorem and basic norm estimates.

6. ERGODIC THEOREMS

6.a. Quantitative uniform ergodic theorems. In this section we discuss Kazhdan projections in the setting of ergodic theory. One of the first ergodic theorems for probability preserving actions of groups was proved by Oseledec [50]. It states that for every locally compact group G and every measure preserving action on a probability space (X, ν) , the powers ρ^k of a density function of a probability measure μ on G form an *ergodic sequence*:

$$\pi(\rho^k)f \longrightarrow \int_X f d\nu(x),$$

both ν -almost everywhere and in $L_p(X, \nu)$. See e.g. [12, 43] for an overview and historical background.

The use of convolutions to construct Kazhdan projections naturally brings us into a setting similar to the Oseledec's ergodic theorem, essential differences being the mode of convergence and the existence of an atomic part for the measures. For Kazhdan projections the convergence in operator norm is

necessary, while the Oseledets theorem gives convergence in the strong operator topology. Therefore the existence of Kazhdan projections can be viewed as a strong ergodic theorem.

An action of a group G on a probability space (X, ν) that is measure preserving induces an action by isometries on any Bochner space $L_p(X, \nu; E)$, where E is a Banach space, by $(\pi_\gamma f)(x) = f(\gamma^{-1}x)$. The Bochner space is uniformly convex provided E is, and $1 < p < \infty$. As a special case of Theorem 1.1, we thus obtain the following quantitative ergodic theorem.

Theorem 1.4. (*The Quantitative Ergodic Theorem*) *Let G be a locally compact group and let $(X_i, \nu_i), i \in I$, be a family of probability spaces endowed with measure preserving ergodic actions of G . Consider also a collection \mathcal{E} of uniformly convex Banach spaces, and a number p in $(1, \infty)$.*

Assume that a family \mathcal{F} of isometric representations of G on $L_p(X_i, \nu; E)$, with $i \in I$ and $E \in \mathcal{E}$, induced by the measure preserving actions of G , has a spectral gap and let (Q, κ) be a Kazhdan pair.

For every Q -admissible measure μ on G there exists $\lambda < 1$, depending only on p , the normalizing factor of μ , the modulus of convexity of \mathcal{E} , and κ , such that

$$(13) \quad \left\| A_\pi^{\mu^k} f - \int_X f d\nu \right\|_p \leq \lambda^k \|f\|_p.$$

Note that if G has property (T) then the family \mathcal{F} composed of all the isometric representations of G on $L_p(X_i, \nu; E)$, for $i \in I$ and E an arbitrary Hilbert space, has a spectral gap.

Proof. Since $\mathcal{P}_\pi = \int_X f d\nu$, the assertion follows from Theorem 1.1 (or, more precisely, from Theorem 3.4). \square

Remark 6.1 (Pointwise convergence). By a standard Borel–Cantelli argument, we deduce from the above that for every measure preserving ergodic action of G on a probability space (X, ν) , for almost every $x \in X$ there exists k_x such that for $k \geq k_x$

$$\left| A_\pi^{\mu^k} f(x) - \int_X f \right| \leq k^{\frac{1+\varepsilon}{p}} \lambda^k \|f\|_p \leq \lambda^{(1-\varepsilon)k} \|f\|_p.$$

In [2] it was shown that for a locally compact group G and every $1 < p < \infty$ property (T) implies property (TL_p) and a Kazhdan pair for the family \mathcal{F} of all isometric representations of G on L_p -spaces can be obtained using p and a Kazhdan pair for the class of all unitary representations of G . As a corollary we obtain the following ergodic theorems for groups with property (T).

Theorem 6.2. *Assume that G has Kazhdan’s property (T). Then the assumptions of Theorem 1.4 hold for every probability preserving action of G , every $1 < p < \infty$ and $E = \mathbb{R}$.*

The uniformity of the convergence is in this case a consequence of property (T). In particular, the convergence is independent of the action. Similar

quantitative ergodic theorems for sequences other than convolution powers were considered by Gorodnik and Nevo, see [23] for a survey and description of applications to Number Theory, as well as the extensive survey [43] on ergodic theorems.

Remark 6.3. Note that for the groups considered by Lafforgue in [35], ergodic theorems for Bochner spaces of a more general type hold. Indeed, consider $G = SL_3(F)$ where F is a non-archimedean local field. The group G has $(T_{\mathcal{E}}^{Ban})$ for the class \mathcal{E} of all representations of small exponential growth on any uniformly convex Banach space [35].

Then two types of ergodic theorems follow. First, by applying Theorem 1.1 we obtain an ergodic theorem for convolution powers of *any* admissible measure on G , as in 1.4 for any uniformly convex Banach space E . Second, it follows directly from property $(T_{\mathcal{E}}^{Ban})$ that for $G = SL_3(F)$ the ergodic theorem as above holds for convolution powers of a *certain* probability measure μ and for actions which are not necessarily measure preserving, but whose Radon-Nikodym derivatives satisfy conditions implying small exponential growth of the corresponding representations. Similar results for $G = Sp_4$ and $G = SL_3(\mathbb{Z})$ are natural corollaries of the results in [38] and [57], respectively.

6.b. Shrinking target problems. The shrinking target problems are formulated for measure preserving ergodic actions of groups on (metric) measure spaces, and are yet another way of understanding such actions. They are particularly significant in the case of actions of subgroups H of Lie groups G on quotients G/Γ , where Γ is a lattice in G . In this case they have Number Theory interpretations as well, especially for G semisimple group. For semisimple Lie groups G and finite volume quotients G/Γ , the shrinking target problems can be classified following the position of the “target”, which can be in the boundary at infinity or inside G/Γ ; or following the type of subgroup H for which the action on G/Γ is considered. Most existing results study actions of amenable subgroups of G (most often, cyclic or one dimensional). The earliest results have been proved for targets at infinity and geodesic flows (i.e. actions of one dimensional subgroups H composed of semisimple elements). The problem of finding the generic behavior of geodesic rays with respect to a shrinking target situated in a cusp was completely settled in the work of Sullivan [62] and Kleinbock and Margulis [32]. The argument in [32] uses theorems of Howe-Moore type and fast decay of correlation coefficients, and the fact that the characteristic function of a neighborhood of a cusp may be replaced by a smooth function, without significant loss of information as far as shrinking target questions are concerned.

When the “target” is not at infinity, but inside G/Γ , the problem becomes that of finding the generic behavior of orbits of H in terms of distance to a fixed point; for instance, of finding the generic speed at which an orbit of H approaches that point. In this case, the methods based on Howe-Moore theorems fail, because characteristic functions of shrinking balls around a point cannot be replaced by smooth functions without a very significant loss

in accuracy. Still, the generic behavior of geodesic rays with respect to a point in simple Lie groups of rank one has been found by D. Sullivan [62] and F. Maucourant [42], using methods specific to rank one. The higher rank case remains open. The question in full generality, of measuring the rate at which a typical orbit approaches a point in G/Γ , has been asked by Kleinbock and Margulis in [32].

Here we show that, as Theorem 1.1 provides a good way to average in a group H with property (T) (average that, in many ways, plays the part of the average on Følner sets for amenable groups), it also allows to answer shrinking target problems for orbits of subgroups H of G that have property (T).

Let (Y, ν) be a probability space. For every integrable function f on Y we denote by Mf its mean, that is $Mf = \int_Y f d\nu$. Let $\Phi = \{f_n : Y \rightarrow \mathbb{R}_+ ; n \in \mathbb{N}\}$ be a sequence of non-negative integrable functions on Y . For $N \in \mathbb{N}$ consider the partial sums of series

$$\Sigma_\Phi^N(y) = \sum_{i=1}^N f_i(y) \quad \text{and} \quad E_\Phi^N = \sum_{i=1}^N Mf_i.$$

Lemma 6.4 ([33], Chapter 1, Lemma 10 in [61]). *Let Y be as above and let p be a positive real number larger than 1.*

- (i) *For μ -almost every $y \in Y$ we have $\liminf_{N \rightarrow \infty} \frac{\Sigma_\Phi^N(y)}{E_\Phi^N} < \infty$.*
- (ii) *Assume that $Mf_n \leq 1$ for every $n \in \mathbb{N}$ and that there exists $C > 0$ such that for every $N > M \geq 1$,*

$$(14) \quad \int_Y \left| \sum_{i=M}^N f_i(y) - \sum_{i=M}^N Mf_i \right|^p d\mu \leq C \sum_{i=M}^N Mf_i.$$

Then for every $\varepsilon > 0$ one has that for μ -almost every $y \in Y$

$$\Sigma_\Phi^N(y) = E_\Phi^N + O\left(\left(E_\Phi^N\right)^{1/p} \log^{1+1/p+\varepsilon}\left(E_\Phi^N\right)\right).$$

Proof. The proof follows verbatim the one of Lemma 10 in [61], except that on top of page 48 one has to apply the Hölder inequality instead of the Cauchy-Schwartz inequality. \square

Let G be a locally compact group, and Γ a lattice in it. We consider G/Γ endowed with the probability measure ν induced by the Haar measure on G , properly renormalized.

We now consider another locally compact group Λ with property (T), S a Kazhdan set of Λ , and μ a probability measure on Λ admissible with respect to S . Assume that Λ acts on G/Γ by transformations preserving ν , and that the action is ergodic.

Examples 6.5. (i) When G is a semisimple group and Γ is an irreducible lattice in G , every infinite subgroup Λ of G acts ergodically on G/Γ [66].

- (ii) When $G = \mathbb{R}^n$, $\Gamma = \mathbb{Z}^n$, with $n \geq 2$, every subgroup Λ of $SL_n(\mathbb{Z})$ containing a matrix whose eigenvalues are not roots of unity acts ergodically on $G/\Gamma = \mathbb{T}^n$ [30, Ex. 4.2.11]

Denote by X_n the random variable representing the n -th step of the random walk on Λ determined by the measure μ . For every $x \in G/\Gamma$ we write $X_n(x)$ to signify the element in G/Γ obtained by applying the group element X_n in Λ to x .

Let p be a positive number larger than 1, and let π be the standard representation of the group Λ by linear isometries on the Banach space $E = L^p(G/\Gamma)$. We consider the action of Λ on E to the right, that is $g \cdot f = f \circ g$. In particular, for $f = \mathbb{1}_\Omega$ the characteristic function of a measurable set Ω , $g \cdot \mathbb{1}_\Omega = \mathbb{1}_{g^{-1}\Omega}$.

Because the action of Λ is ergodic, the space E^π is composed of constant functions, while E_π is composed of functions of the form $f - Mf$, for every $f \in L^p(G/\Gamma)$ (due to the fact that the dual of $L^p(G/\Gamma)$ can be identified with $L^q(G/\Gamma)$, where $\frac{1}{p} + \frac{1}{q} = 1$). As Λ has property (T), it follows, by Theorem 1.1 and [2, Theorem A], that there exists $\lambda \in (0, 1)$ depending on the Kazhdan constant of S and on p , such that $\|A_\pi^\mu|_{E_\pi}\| \leq \lambda$.

Let h be a non-negative integrable function on G/Γ . For an arbitrary probability measure α on Λ , if we consider the function $f = A_\pi^\alpha(h)$ then by Fubini's Theorem we can write

$$\begin{aligned} M(f) &= \int_{G/\Gamma} f(x) d\nu(x) = \int_{G/\Gamma} \left[\int_\Lambda h \circ g(x) d\alpha(g) \right] d\nu(x) \\ &= \int_\Lambda \left[\int_{G/\Gamma} h(gx) d\nu(x) \right] d\alpha(g) = M(h). \end{aligned}$$

In particular, the above is true for the function $f = A_\pi^{\mu^n}(\mathbb{1}_\Omega)$, where Ω is a measurable set in G/Γ . Note that for $x \in G/\Gamma$, we have that

$$f(x) = \mathbb{P}(X_n(x) \in \Omega).$$

Consider now a sequence $(\Omega_n)_{n \in \mathbb{N}}$ of measurable sets in G/Γ , and the sequences of measurable functions $h_n = \mathbb{1}_{\Omega_n}$ and $f_n = A_\pi^{\mu^n}(\mathbb{1}_{\Omega_n})$. We prove that the hypotheses of Lemma 6.4 are satisfied. The left hand side of the inequality (14) can be bounded as follows

$$\begin{aligned} \left\| \sum_{i=M}^N \left(A_\pi^{\mu^i} h_i - M h_i \right) \right\|_p &\leq \sum_{i=M}^N \lambda^i \|h_i - M h_i\|_p \\ &\leq \left(\sum_{i=M}^N \lambda^{qi} \right)^{1/q} \left(\sum_{i=M}^N \|h_i - M h_i\|_p^p \right)^{1/p}. \end{aligned}$$

The first inequality above uses the property that on the subspace E_π , composed of functions of the form $f - Mf$, the norm of $A_\pi^{\mu^i}$ is bounded by λ^i , the second uses the Hölder inequality.

Elementary calculations yield the following inequality (see for instance [20, §4])

$$|a - b|^p \leq a^p + b^p + (1 + p2^p) \max(a^{p-1}b, ab^{p-1}) \text{ for every } a, b \in \mathbb{R}_+.$$

This allows us to write

$$\int |h_i - Mh_i|^p d\nu(x) \leq \int h_i^p d\nu(x) + Mh_i^p + C_p Mh_i \int h_i^{p-1} d\nu(x) \leq (2 + C_p) Mh_i,$$

where $C_p = 1 + p2^p$. In the last inequality above we used the facts that h_i is the characteristic function of a set, hence $h_i^\alpha = h_i$ for every $\alpha \geq 1$, and that $Mh_i \in (0, 1)$, therefore $Mh_i^\beta \leq Mh_i$ for every $\beta \geq 1$.

We may therefore write

$$\left\| \sum_{i=M}^N \left(A_\pi^{\mu_i} h_i - Mh_i \right) \right\|_p^p \leq \frac{2 + C_p}{(1 - \lambda^q)^{p/q}} \sum_{i=M}^N Mh_i.$$

Lemma 6.4 then implies the following.

Theorem 6.6. *Let G be a locally compact group, and Γ a lattice in it. Let $\{\Omega_n\}$ be a sequence of measurable subsets in G/Γ .*

Assume that a locally compact group Λ with property (T) acts ergodically on G/Γ . Let μ be a probability measure on Λ that is admissible with respect to a Kazhdan set, and let X_n be the random variable representing the n -th step of the random walk defined by μ .

(i) *If $\sum_n \nu(\Omega_n)$ is finite then for almost every $x \in G/\Gamma$*

$$\sum_{n \in \mathbb{N}} \mathbb{P}(X_n(x) \in \Omega_n) < \infty.$$

(ii) *If $\sum_n \nu(\Omega_n)$ is infinite then for every $\varepsilon > 0$ and for almost every $x \in G/\Gamma$,*

$$\sum_{n \leq N} \mathbb{P}(X_n(x) \in \Omega_n) = S_N + O(S_N^\varepsilon),$$

where $S_N = \sum_{n \leq N} \nu(\Omega_n)$. In particular, $\mathbb{P}(X_n(x) \in \Omega_n) > 0$ for infinitely many $n \in \mathbb{N}$.

Corollary 6.7. *Let G be a Lie group, and Γ , Λ and X_n as above. Let dist be a distance on G/Γ induced by a left invariant Riemannian distance on G . Let x_0 be an arbitrary point in G/Γ , let D be the dimension of G/Γ and let $\{r_n\}_{n \in \mathbb{N}}$ be a sequence of positive numbers converging to 0.*

(i) *If $\sum_n r_n^D$ is finite then for almost every $x \in G/\Gamma$*

$$\sum_{n \in \mathbb{N}} \mathbb{P}(\text{dist}(X_n(x), x_0) \leq r_n) < \infty.$$

(ii) *Assume that $\sum_n r_n^D$ is infinite. Then for every $\varepsilon > 0$, almost every $x \in G/\Gamma$ satisfies*

$$\sum_{n \leq N} \mathbb{P}(\text{dist}(X_n(x), x_0) \leq r_n) = R_N + O(R_N^\varepsilon),$$

where $R_N = \sum_{n \leq N} r_n^D$.

In particular, $\mathbb{P}(\text{dist}(X_n(x), x_0) \leq r_n) > 0$ for infinitely many $n \in \mathbb{N}$.

Since Λ has property (T), it is compactly generated. Let dist_Λ be an arbitrary word metric on it, corresponding to the choice of a compact generating set. Let α be a probability measure on Λ with compact support generating Λ as a semigroup, and let X_n be the corresponding random walk. Since Λ is non-amenable, according to [31, 24] there exists $a > 0$ such that almost surely

$$\lim_{n \rightarrow \infty} \frac{\text{dist}_\Lambda(e, X_n)}{n} = 2a.$$

This implies that

$$\mu^n(\Delta_n) \geq 1 - \varepsilon_n,$$

where $\Delta_n = \{g \in \Lambda : \text{dist}_\Lambda(e, g) > an\}$ and $\lim_{n \rightarrow \infty} \varepsilon_n = 0$.

Consider the sequence of probability measures $\eta_n = \frac{1}{\mu^n(\Delta_n)} \mu^n|_{\Delta_n}$. The previous argument, with the sequence μ^n replaced by η_n , gives the following.

Corollary 6.8. *Let G be a locally compact group, and Γ a lattice in it. Assume that a locally compact group Λ with property (T) acts ergodically on G/Γ . Let dist_Λ be a word metric on Λ , and let X_n be the random variable representing the n -th step of a random walk on Λ corresponding to an admissible measure. There exists a constant $a > 0$ such that the following holds.*

- (i) *If a sequence $\{\Omega_n\}$ of measurable subsets in G/Γ is such that $\sum_n \nu(\Omega_n)$ is infinite then for almost every $x \in G/\Gamma$,*

$$\sum_{n \leq N} \mathbb{P}(X_n(x) \in \Omega_n, \text{dist}_\Lambda(X_n, e) \geq an) = S_N + o(S_N),$$

where $S_N = \sum_{n \leq N} \nu(\Omega_n)$.

- (ii) *Assume moreover that G is a Lie group, and dist is a distance on G/Γ induced by a left invariant Riemannian distance on G .*

If x_0 is an arbitrary point in G/Γ , D is the dimension of G/Γ and $\{r_n\}_{n \in \mathbb{N}}$ is a sequence of positive numbers such that $\sum_n r_n^D$ is infinite, then for almost every $x \in G/\Gamma$,

$$\sum_{n \leq N} \mathbb{P}(\text{dist}(X_n(x), x_0) \leq r_n, \text{dist}_\Lambda(e, X_n) \geq an) = R_N + o(R_N),$$

where $R_N = \sum_{n \leq N} r_n^D$.

7. GHOST PROJECTIONS FOR WARPED CONES

In this section we construct new examples of non-compact ghost projections and answer a question of Willett and Yu [64, Problem 5.4]. The existence of such projections in the case of a sequence of expanders arising from an exact group with property (T) or (τ) [64, Examples 5.3] is the only known reason for the failure of the coarse Baum-Connes conjecture [26, 27, 64]. In fact, no examples of ghost projections other than the one described in [26], [64, Examples 5.3] were known until now.

Note that for our construction the acting group has to be neither residually finite nor with properties (T) or (τ) – the existence of the ghost projection is solely a consequence of the spectral gap of the action. A spectral gap for

a single representation is a weaker condition than property (τ) . There are many examples of actions with a spectral gap: see for instance [43, Example 11.3] for actions of non-amenable algebraic groups, and [19] for actions on tori and nil-manifolds. Clearly, if G has property (T) then every probability preserving action of G has a spectral gap.

7.a. Warped cones as metric measure spaces. Let (M, dist, m) be a compact metric space endowed with a probability measure, and let G be a finitely generated group acting on M by bi-Lipschitz homeomorphisms preserving the measure. Assume moreover that the action of G is ergodic. Throughout this section we consider G endowed with a finite, symmetric generating set S . We denote by $|g|$ the word length with respect to S of $g \in G$.

We add a mild assumption on the measure, requiring in some sense its uniform distribution with respect to the metric.

Definition 7.1. *A measure m on a metric space (M, dist) is called upper uniform if $\lim_{R \rightarrow 0} \sup_{x \in M} m(B(x, R)) = 0$.*

Let $\text{Cone}(M) = M \times (1, \infty)$ denote the truncated Euclidean cone over M , equipped with the measure ν which is a product measure of m and the Lebesgue measure. The restriction of the metric $\text{dist}_{\text{Cone}(M)}$ on $\text{Cone}(M)$ to $M \times \{t\}$ is equal to $t \text{dist}$. The part corresponding to the interval $[0, 1]$, both for the Euclidean cone, and for the warped cone defined below, is irrelevant for our purposes, since we are interested in large scale properties, and removing this part simplifies certain estimates.

The notion of a *warped cone*, denoted by $\mathcal{O} = \mathcal{O}_G(M)$, was first defined in [55]. It is the space $M \times (1, \infty)$ endowed with the metric $\text{dist}_{\mathcal{O}}$ that is the largest metric satisfying

$$\text{dist}_{\mathcal{O}}(x, y) \leq \text{dist}_{\text{Cone}(M)}(x, y) \quad \text{and} \quad \text{dist}_{\mathcal{O}}(x, sx) \leq 1,$$

for every $x, y \in \mathcal{O}$ and $s \in S$. We endow \mathcal{O} with the product measure ν of m and the Lebesgue measure.

For $t \geq 1$ we denote by \mathcal{O}_t the part of the warped cone \mathcal{O} that corresponds to $M \times (t, \infty)$.

In [55] it is shown that the warped metric from x to y is the infimum over all sums

$$(15) \quad \sum_{i=0}^{k-1} \text{dist}(g_i x_i, x_{i+1}) + |g_i|,$$

taken over finite sequences $x = x_0, x_1, \dots, x_k = y$ in M and g_0, \dots, g_{k-1} in G . Moreover, if $\text{dist}(x, y)_{\mathcal{O}} \leq n$, where $n \in \mathbb{N}$, then we can choose $k \leq n + 1$. Thus a warped cone is a metric space that interpolates between orbits of the action at $t = 1$ and the group G with the word metric at $t = \infty$.

If $\text{Cone}(M)$ has bounded geometry (e.g. M embeds into a finite-dimensional Euclidean space) (see [45, 54]) and G acts on $\text{Cone}(M)$ by bi-Lipschitz homeomorphisms then $\mathcal{O}_G(M)$ has bounded geometry [55].

The following statement describes the relation of a ball in the warped metric to a ball in the Euclidean cone.

Lemma 7.2. *Assume that G acts on M by bi-Lipschitz maps. Then for each $R > 0$ there exists $T > 0$ such that every ball of radius R in the warped cone \mathcal{O} , with center an arbitrary point $x \in \mathcal{O}$, satisfies*

$$B_{\mathcal{O}}(x, R) \subseteq \bigcup_{|g| \leq R} B_{\text{Cone}(M)}(gx, T).$$

Proof. We use the definition of the warped metric as infimum of finite sums of the form described in (15). Consider the case $g_0 = e$. The other case is proved analogously and is omitted. We have $x_1 \in B(x_0, R)$. The next step in the chain is realized by $g_1 x_1$, for some group element $g_1 \in B_G(e, R_1)$, where in this case we set $R_1 = R$. Then $x_2 \in X$ is such that the inequality

$$\text{dist}(x_2, g_1 x_1) \leq R_2 = R - \text{dist}(x_0, x_1) - |g_1|$$

is satisfied. Observe now that

$$\begin{aligned} \text{dist}(g_1 x_0, x_2) &\leq \text{dist}(g_1 x_0, g_1 x_1) + \text{dist}(g_1 x_1, x_2) \\ &\leq L_{g_1} \text{dist}(x_0, x_1) + R_2 \\ &\leq L_{g_1} R_1 + R_2, \end{aligned}$$

so that $x_2 \in B(g_1 x_0, L_{g_1} R_1 + R_2)$, where L_g denotes the Lipschitz constant of the transformation of M associated to g . Following these estimates for g_2 and x_3 we observe that $x_3 \in B(g_2 x_0, L_{g_2 g_1} R_1 + L_{g_2} R_2 + R_3)$. In general, for every $i = 0, 1, \dots, k$ there exists $T(g_i)$ such that $x_{i+1} \in B_{\text{Cone}(M)}(g_i x, T(g_i))$, where the radii $T(g_i)$ depend on R and the Lipschitz constants of the transformations of M associated to g_i , but can be chosen independently of x . All possible choices for g_i have to satisfy $|g_i| \leq R$. Therefore setting $T = \sup\{T(g) : |g| \leq R\}$ we obtain the claim. \square

Note that if the action of G on M is isometric then we can take $T = R$ in Lemma 7.2.

Lemma 7.3. *For every $R, \varepsilon > 0$ there exists $t \in (1, \infty)$ such that*

$$v(B_{\mathcal{O}}(x, R)) \leq \varepsilon,$$

for every $x \in \mathcal{O}_t$.

Proof. Let $T > 0$ and $x = (y, s) \in \mathcal{O}_t$. Every ball $B_{\text{Cone}(M)}(x, T)$ is contained in a product $B(y, r) \times [s - T, s + T]$, where r can be chosen with an upper bound depending only on T and s . Then

$$v(B_{\text{Cone}(M)}(x, T)) \leq 2Tm(B(y, r)),$$

which tends to zero uniformly as $t \rightarrow \infty$, by the upper uniformity of the measure m . By the previous lemma we also have

$$v(B_{\mathcal{O}}(x, R)) \leq \sum_{|g| \leq R} v(B_{\text{Cone}(M)}(gx, T)) \leq 2Tm(B(y, r))|B_G(e, R)|,$$

which again tends to zero uniformly when $t \rightarrow \infty$, as R and T are fixed. \square

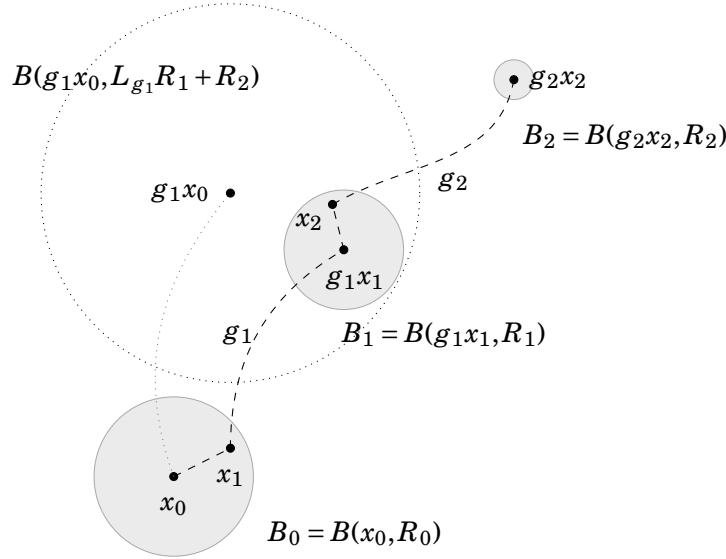


FIGURE 1. A possible path realizing a distance $\leq R$ in the warped metric. The length of this path is $\text{dist}(x_0, x_1) + |g_1| + \text{dist}(g_1 x_1, x_2) + |g_2| \leq R$.

7.b. Finite propagation operators on a warped cone. Consider a locally compact metric measure space (X, dist, m) . An X -module is a (separable) Hilbert space H equipped with a representation of $C_0(X)$. An operator $T \in B(H)$ has *finite propagation* if there exists $S > 0$ such that for $\phi, \psi \in C_0(X)$ satisfying $\text{dist}(\text{supp } \phi, \text{supp } \psi) > S$ we have $\phi T \psi = 0$.

The space $H = L_2(\mathcal{O}, \nu)$ is equipped with the standard, faithful representation of $C_0(\mathcal{O})$ by multiplication operators and thus naturally becomes an \mathcal{O} -module.

The action of G on \mathcal{O} preserves the measure ν and induces a unitary representation π of G on $L_2(\mathcal{O}, \nu) \simeq L_2(M \times (1, \infty))$ defined by $\pi_g f(x, t) = f(g^{-1}x, t)$.

Proposition 7.4. *For every $g \in G$ the operator π_g has bounded propagation on $L_2(\mathcal{O}, \nu)$.*

In particular, the Markov operator A_π^μ has bounded propagation for any choice of a probability measure μ supported on a finite generating set S of G .

Proof. Assume that $\phi \pi_g \psi \neq 0$ for $\phi, \psi \in C_0(\mathcal{O})$. This is possible only if $\text{supp } \phi \cap g^{-1}(\text{supp } \psi) \neq \emptyset$. This however implies that $\text{dist}_{\mathcal{O}}(\text{supp } \phi, \text{supp } \psi) \leq |g|$. \square

7.c. Noncompact ghost projections. The notion of ghost operator was introduced by G. Yu (unpublished).

Definition 7.5. *Given a metric measure space (X, dist, ν) , an operator $T \in \mathcal{B}(L_2(X, \nu))$ is said to be a ghost if for every $R, \varepsilon > 0$ there exists a bounded set*

$B \subseteq X$ such that for $\psi \in L_2(X, \nu)$ satisfying $\|\psi\| = 1$ and $\text{supp } \psi \subseteq B(x, R)$ for some $x \in X \setminus B$ we have $\|T\psi\| \leq \varepsilon$.

Ghost operators are operators that are *locally invisible at infinity* [15]. Such operators are intrinsically connected to large scale geometric features of the space [15, 16, 56]. Non-compact ghost projections are central objects in the context of the Baum-Connes conjecture, see below. We refer to [26, 27, 54, 64] for a detailed discussion.

Define $\mathfrak{G} : L_2(\mathcal{O}, \nu) \rightarrow L_2(\mathcal{O}, \nu)$ by setting

$$\mathfrak{G}f(x, t) = \int_{M \times \{t\}} f(y, t) dm(y)$$

for every $(x, t) \in M \times (1, \infty)$. The map \mathfrak{G} is the orthogonal projection onto the subspace V composed of functions that are constant on $M \times \{t\}$ for every $t \in (1, \infty)$. The subspace V is a copy of $L_2(1, \infty)$ embedded in $L_2(\mathcal{O}, \nu)$.

Theorem 7.6. *Let (M, dist, m) be a compact metric space endowed with a probability measure, and let G be a finitely generated group acting on M ergodically by bi-Lipschitz homeomorphisms preserving the measure. If the action of G on M has a spectral gap then $\mathfrak{G} \in \mathcal{B}(L_2(\mathcal{O}, \nu))$ is a non-compact ghost projection that is a norm limit of finite propagation operators.*

Proof. The projection \mathfrak{G} is not compact since its range is infinite-dimensional. The action has a spectral gap therefore, by Theorems 1.1 and 3.8, for every probability measure μ on G admissible with respect to a finite generating set and of density function ρ , there exists $\lambda < 1$ such that the following holds. For every $t > 1$ we have

$$\left\| \pi(\rho^k) f|_{M \times \{t\}} - \mathfrak{G}f(\cdot, t) \right\|_{L_2(M, m)}^2 \leq \lambda^{2k} \|f|_{M \times \{t\}}\|_{L_2(M, m)}^2.$$

Using Fubini's theorem we conclude that

$$\left\| \pi(\rho^k) - \mathfrak{G} \right\|_{\mathcal{B}(L_2(\mathcal{O}, \nu))} \leq \lambda^k,$$

where $\pi(\rho^k) = (A_\pi^\mu)^k$ has finite propagation by Proposition 7.4.

It remains to show that \mathfrak{G} is a ghost. Let $R, \varepsilon > 0$. For every $\delta > 0$ there exists $t > 0$ such that $\nu(B_{\mathcal{O}}(p, R)) \leq \delta$ for every $p \in \mathcal{O}_t$, by lemma 7.3. Consider $f \in L_2(\mathcal{O}, \nu)$ such that $\text{supp } f \subseteq B(p, R)$ for some $p \in \mathcal{O}_t$. For every $s \in (t, \infty)$ the projection \mathfrak{G} satisfies

$$\begin{aligned} \mathfrak{G}f(x, s)^2 &= \left(\int_{M \times \{s\}} f(y, s) dm(y) \right)^2 \\ &= m(\text{supp } f \cap (M \times \{s\}))^2 \left(\frac{1}{m(\text{supp } f \cap (M \times \{s\}))} \int_{\text{supp } f \cap M \times \{s\}} f(y, s) dm(y) \right)^2 \\ &\leq m(\text{supp } f \cap (M \times \{s\}))^2 \int_{\text{supp } f \cap M \times \{s\}} f(y, s)^2 dm(y) \\ &\leq m(\text{supp } f \cap (M \times \{s\}))^2. \end{aligned}$$

The above implies, by integration and Fubini's theorem, that $\|\mathfrak{G}f\|^2 \leq \nu(\text{supp } f)^2 \leq \nu(B_{\mathcal{O}}(p, R))^2 \leq \delta^2$. \square

We point out that although Theorem 7.6 is formulated for Hilbert spaces, the same proof gives a straightforward stronger version. Namely, provided that the action on the Bochner space $L_2(M, m; E)$ has a spectral gap we obtain that the projection from $L_2(\mathcal{O}, \nu; E)$ onto $L_2([1, \infty); E)$ is a non-compact ghost projection which is a limit of finite propagation operators. At present, however, ghost projections on non-Hilbert spaces do not have applications similar to the ones on Hilbert spaces. We also point out that very likely, the construction admits a generalization to a foliated versions of the warped cone [53].

There are many group actions to which the above theorem applies, in particular actions on compact groups of finitely generated (free) subgroups. This latter type of constructions were motivated by the Ruziewicz problem (see [7, 8] and references therein). Thus, Theorem 7.6 applies for instance to certain warped cones $\mathcal{O}_{\mathbb{F}_n}(\text{SU}(2))$, where the free group \mathbb{F}_n is a subgroup of $\text{SU}(2)$ generated by elements of a specific type [7].

7.d. The coarse Baum-Connes conjecture. The Roe algebra $C^*(X)$ of a space X is the closure of locally compact finite propagation operators. An operator T on an X -module H is *locally compact* if for every $f \in C_0(X)$ the operators fT and Tf are compact. Recall that the coarse Baum-Connes conjecture predicts that for a metric space X of bounded geometry, the coarse assembly map

$$(16) \quad \lim_{d \rightarrow \infty} K_*(P_d(X)) \longrightarrow K_*(C^*(X)),$$

from the coarse K -homology of X to the K -theory of the Roe algebra, is an isomorphism. Above, $P_d(X)$ is the Rips complex at scale $d \geq 0$. If true for a finitely generated group G , the coarse Baum-Connes conjecture implies the Novikov conjecture on the homotopy invariance of higher signatures.

Counterexamples to the coarse Baum-Connes conjecture were constructed in [26, 27]. It was proved there that the coarse assembly map is not surjective for X a coarse disjoint union of expanders obtained as quotients of a group with property (T). The reason is the existence of a non-compact ghost projection \mathfrak{G} which is a limit of finite propagation operators. The K -theory class represented by $\mathfrak{G} \otimes p$ in $K_*(C^*(X))$, where p is a rank one projection, is not in the image of the coarse assembly map. At present such expanders provide the only known counterexamples to the coarse Baum-Connes conjecture.

Let G, M satisfy the assumptions of Theorem 7.6.

Conjecture 7.7. *The coarse assembly map (16) is not surjective for the warped cone $X = \mathcal{O}_G M$.*

REFERENCES

- [1] C. A. Akemann and M. E. Walter, *Unbounded negative definite functions*, Canad. J. Math. **33** (1981), no. 4, 862–871.
- [2] U. Bader, A. Furman, T. Gelander, and N. Monod, *Property (T) and rigidity for actions on Banach spaces*, Acta Math. **198** (2007), no. 1, 57–105.
- [3] U. Bader and P. W. Nowak, *Cohomology of deformations*, J. Topol. Anal. **7** (2015), no. 1, 81–104.
- [4] U. Bader, C. Rosendal, and R. Sauer, *On the cohomology of weakly almost periodic group representations*, J. Topol. Anal. **6** (2014), no. 2, 153–165.
- [5] B. Bekka and B. Olivier, *On groups with property (T_{ℓ_p})* , J. Funct. Anal. **267** (2014), no. 3, 643–659.
- [6] B. Bekka, P. de la Harpe, and A. Valette, *Kazhdan’s property (T)*, New Mathematical Monographs, vol. 11, Cambridge University Press, Cambridge, 2008.
- [7] J. Bourgain and A. Gamburd, *On the spectral gap for finitely-generated subgroups of $SU(2)$* , Invent. Math. **171** (2008), no. 1, 83–121.
- [8] J. Bourgain and A. Gamburd, *A spectral gap theorem in $SU(d)$* , J. Eur. Math. Soc. (JEMS) **14** (2012), no. 5, 1455–1511.
- [9] J. Bourgain and P. Varjú, *Expansion in $SL_d(\mathbf{Z}/q\mathbf{Z})$, q arbitrary*, Invent. Math. **188** (2012), no. 1, 151–173.
- [10] M. Bourdon and H. Pajot, *Cohomologie l_p et espaces de Besov*, J. Reine Angew. Math. **558** (2003), 85–108.
- [11] N. P. Brown, *Kazhdan’s property T and C^* -algebras*, J. Funct. Anal. **240** (2006), no. 1, 290–296.
- [12] A. Bufetov and A. Klimenko, *On Markov operators and ergodic theorems for group actions*, European J. Combin. **33** (2012), no. 7, 1427–1443.
- [13] M. Burger, *Kazhdan constants for $SL(3, \mathbf{Z})$* , J. Reine Angew. Math. **413** (1991), 36–67.
- [14] I. Chatterji, C. Druțu, and F. Haglund, *Kazhdan and Haagerup properties from the median viewpoint*, Adv. Math. **225** (2010), no. 2, 882–921.
- [15] X. Chen and Q. Wang, *Ideal structure of uniform Roe algebras of coarse spaces*, J. Funct. Anal. **216** (2004), no. 1, 191–211.
- [16] X. Chen and Q. Wang, *Ghost ideals in uniform Roe algebras of coarse spaces*, Arch. Math. (Basel) **84** (2005), no. 6, 519–526.
- [17] F. Chong, *E-theory for L^p -algebras and the dual Novikov conjecture*, PhD thesis, Vanderbilt University (2014).
- [18] A. Connes, *Noncommutative geometry*, Academic Press, Inc., San Diego, CA, 1994.
- [19] J.-P. Conze and Y. Guivarc’h, *Ergodicity of group actions and spectral gap, applications to random walks and Markov shifts*, Discrete Contin. Dyn. Syst. **33** (2013), no. 9, 4239–4269.
- [20] C. Druțu and J. Mackay, *Random groups, random graphs and eigenvalues of p -Laplacians*, preprint.

- [21] D. Fisher and G. Margulis, *Almost isometric actions, property (T), and local rigidity*, Invent. Math. **162** (2005), no. 1, 19–80.
- [22] E. Gardella and H. Thiel, *Group algebras acting on L^p -spaces*, arXiv:1408.6136 [math.FA].
- [23] A. Gorodnik and A. Nevo, *Quantitative ergodic theorems and their number-theoretic applications*, Bull. Amer. Math. Soc. (N.S.) **52** (2015), no. 1, 65–113.
- [24] Y. Guivarc’h, *Sur la loi des grands nombres et le rayon spectral d’une marche aléatoire*, Conference on Random Walks (Kleebach, 1979), 1980, pp. 47–98, 3.
- [25] Pierre and Valette de la Harpe Alain, *La propriété (T) de Kazhdan pour les groupes localement compacts (avec un appendice de Marc Burger)*, 1989. With an appendix by M. Burger.
- [26] N. Higson, *Counterexamples to the coarse Baum-Connes conjecture* (1999).
- [27] N. Higson, V. Lafforgue, and G. Skandalis, *Counterexamples to the Baum-Connes conjecture*, Geom. Funct. Anal. **12** (2002), no. 2, 330–354.
- [28] N. Higson and J. Roe, *Analytic K-homology*, Oxford Mathematical Monographs, Oxford University Press, Oxford, 2000. Oxford Science Publications.
- [29] G. Kasparov, *On the L^p Novikov and Baum-Connes conjectures*, <https://www.math.kyoto-u.ac.jp/kida/conf/ask2013/Kasparov.pdf>.
- [30] A. Katok and B. Hasselblatt, *Introduction to the modern theory of dynamical systems*, Encyclopedia of Mathematics and its Applications, vol. 54, Cambridge University Press, Cambridge, 1995.
- [31] H. Kesten, *Full Banach mean values on countable groups*, Math. Scand. **7** (1959), 146–156.
- [32] D. Kleinbock and Margulis G. A., *Logarithm laws for flows on homogeneous spaces*, Invent. math. **138** (1999), 451–494.
- [33] S. Kochen and C. Stone, *A note on the Borel Cantelli Lemma*, Ill. J. math. **8** (1964), 248–251.
- [34] T. de Laat and M. de la Salle, *Noncommutative- L^p -rigidity for high rank lattices and nonembeddability of expanders*, arXiv:1403.6415 [math.OA].
- [35] V. Lafforgue, *Un renforcement de la propriété (T)*, Duke Math. J. **143** (2008), no. 3, 559–602.
- [36] V. Lafforgue, *Propriété (T) renforcée banachique et transformation de Fourier rapide*, J. Topol. Anal. **1** (2009), no. 3, 191–206.
- [37] V. Lafforgue, *Propriété (T) renforcée et conjecture de Baum-Connes*, Quanta of maths, Clay Math. Proc., vol. 11, Amer. Math. Soc., Providence, RI, 2010, pp. 323–345.
- [38] B. Liao, *Strong Banach property (T) for simple algebraic groups of higher rank*, J. Topol. Anal. **6** (2014), no. 1, 75–105.
- [39] A. Lubotzky, *Discrete groups, expanding graphs and invariant measures*, Progress in Mathematics, vol. 125, Birkhäuser Verlag, Basel, 1994.

- [40] A. Lubotzky, *What is...property (τ) ?*, Notices Amer. Math. Soc. **52** (2005), no. 6, 626–627.
- [41] A. Lubotzky and A. Żuk, *On property (τ)* , available on the first author's website (2003).
- [42] F. Maucourant, *Dynamical Borel-Cantelli lemma for hyperbolic spaces*, Israel J. Math. **152** (2006), 143–155.
- [43] A. Nevo, *Pointwise ergodic theorems for actions of groups*, Handbook of dynamical systems. Vol. 1B, Elsevier B. V., Amsterdam, 2006, pp. 871–982.
- [44] B. Nica, *Proper isometric actions of hyperbolic groups on L^p -spaces*, Compos. Math. **149** (2013), no. 5, 773–792.
- [45] P. W. Nowak and G. Yu, *Large scale geometry*, EMS Textbooks in Mathematics, European Mathematical Society (EMS), Zürich, 2012.
- [46] P. W. Nowak, *Poincaré inequalities and rigidity for actions on Banach spaces*, J. Eur. Math. Soc. (JEMS) **17** (2015), no. 3, 689–709.
- [47] P. W. Nowak, *Group Actions on Banach Spaces*, Handbook of Group Actions, Adv. Lect. Math. (ALM), vol. 32, Int. Press, Somerville, MA, 2015, pp. 121–149.
- [48] Mathematisches Forschungsinstitut Oberwolfach, *Report no. 29/2001. Mini Workshop, Geometrization of Property (T)* .
- [49] H. Oh, *Uniform pointwise bounds for matrix coefficients of unitary representations and applications to Kazhdan constants*, Duke Math. J. **113** (2002), no. 1, 133–192.
- [50] V. I. Oseledets, *Markov chains, skew products and ergodic theorems for “general” dynamic systems*, Teor. Veroyatnost. i Primenen. **10** (1965), 551–557.
- [51] N. C. Phillips, *Open problems related to operator algebras on L^p -spaces*, <http://www.math.ksu.edu/events/conference/gpots2014/LpOpAlgQuestions.pdf>.
- [52] M. Puschnigg, *The Baum-Connes conjecture with coefficients for word-hyperbolic groups (after Vincent Lafforgue)*, Bourbaki seminar 1062, October 2012. arXiv:1211.6009 [math.KT] (2012).
- [53] J. Roe, *From foliations to coarse geometry and back*, Analysis and geometry in foliated manifolds (Santiago de Compostela, 1994), World Sci. Publ., River Edge, NJ, 1995, pp. 195–205.
- [54] J. Roe, *Lectures on coarse geometry*, University Lecture Series, vol. 31, American Mathematical Society, Providence, RI, 2003.
- [55] J. Roe, *Warped cones and property A*, Geom. Topol. **9** (2005), 163–178 (electronic).
- [56] J. Roe and R. Willett, *Ghostbusting and property A*, J. Funct. Anal. **266** (2014), no. 3, 1674–1684.
- [57] M. de la Salle, *Towards Strong Banach property (T) for $SL(3, R)$* , Israel Journal of Mathematics, to appear.
- [58] Y. Shalom, *Expander graphs and amenable quotients*, Emerging applications of number theory (Minneapolis, MN, 1996), 1999, pp. 571–581.

- [59] Y. Shalom, *Explicit Kazhdan constants for representations of semisimple and arithmetic groups*, Ann. Inst. Fourier (Grenoble) **50** (2000), no. 3, 833–863.
- [60] Y. Shalom, *Invariant measures for algebraic actions, Zariski dense subgroups and Kazhdan’s property (T)*, Trans. Amer. Math. Soc. **351** (1999), no. 8, 3387–3412.
- [61] V. Sprindzuk, *Metric theory of Diophantine approximations*, John Wiley & Sons, New York, Toronto, London, 1979.
- [62] D. Sullivan, *Disjoint spheres, approximation by imaginary quadratic numbers, and the logarithm law for geodesics*, Acta Math. **149** (1982), 215–237.
- [63] A. Valette, *Minimal projections, integrable representations and property (T)*, Arch. Math. (Basel) **43** (1984), no. 5, 397–406.
- [64] R. Willett and G. Yu, *Higher index theory for certain expanders and Gromov monster groups, I*, Adv. Math. **229** (2012), no. 3, 1380–1416.
- [65] G. Yu, *Hyperbolic groups admit proper affine isometric actions on l^p -spaces*, Geom. Funct. Anal. **15** (2005), no. 5, 1144–1151.
- [66] R. J. Zimmer, *Ergodic theory and semisimple groups*, Monographs in Mathematics, vol. 81, Birkhäuser Verlag, Basel, 1984.

C.Druţu: MATHEMATICAL INSTITUTE, UNIVERSITY OF OXFORD, UNITED KINGDOM.
E-mail address: Cornelia.Drutu@maths.ox.ac.uk

P. W. Nowak: INSTITUTE OF MATHEMATICS OF THE POLISH ACADEMY OF SCIENCES,
POLAND – AND – INSTITUTE OF MATHEMATICS, UNIVERSITY OF WARSAW, POLAND
E-mail address: pnowak@impan.gov.pl